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A CONVENIENT SETTING FOR DIFFERENTIAL CALCULUS*

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Cartesian closed categories play an important rôle in many aspects of mathematics. They appear in algebraic geometry, in logic, in topology. Quite naturally one is tempted to use this notion also for differential calculus.

First attempts in this direction were undertaken by A. Bastiani [1] and then by A. Frölicher and W. Bucher [4]. They all used the notion of limit spaces for their generalizations of differential calculus. In retrospect one may say that A. Bastiani used a "good" definition of differentiability but the category she chose did not allow cartesian closedness. A. Frölicher and W. Bucher took cartesian closedness exactly as their goal. But because of their "bad" definition of continuous differentiability, their special types of limit vector spaces became increasingly complicated.

Because questions of continuity and differentiability are local and topological spaces have by definition a local structure, it has many advantages to establish differential calculus in a "pure" topological setting. And then the question is: Can a differential calculus be so established in a topological setting as to obtain cartesian closedness in the infinitely often differentiable case?

For a long time it even seemed impossible to obtain anything like cartesian closedness for continuous maps – not to speak of differentiable ones. But in 1963 Gabriel and Zisman proved [6] that the category $\mathscr{C}\mathscr{G}$ of compactly generated hausdorff spaces is cartesian closed and this is a full subcategory of the category of topological spaces.

Starting from this category $\mathscr{C}\mathscr{G}$, we are quite naturally led to investigate the category $\mathscr{C}\mathscr{G}\mathscr{V}$ of compactly generated vector spaces. Observing that the Hahn–Banach theorem is the tool for the proof of the so-called mean value theorem of differential calculus, we see that not all compactly generated vector spaces are convenient for a differential calculus. But there is a nice full subcategory of $\mathscr{C}\mathscr{G}\mathscr{V}$ and the objects of this subcategory will provide our convenient setting.

* *Note added in proof.* The problem of smooth manifolds will be treated in a forthcoming publication in this journal. Using a different point of view, but also based on this setting for differential calculus, cartesian closedness will be obtained in general.

Our notion of (continuous) differentiability is then the simplest one possible: We say that a map $\alpha : E \supset U \to F$ – where E and F are convenient vector spaces and U is open in E – is (continuously) differentiable if there exists a continuous map $\mathrm{d}\alpha : E \sqcap E \supset U \sqcap E \to F$ such that $\mathrm{d}\alpha$ is linear in the second variable and such that for arbitrary $(x, y) \in U \sqcap E$ the equation

$$\mathrm{d}\alpha(x, y) = \lim_{0 \neq t \to 0} \frac{1}{t}\{\alpha(x + ty) - \alpha(x)\}$$

holds. Because the category $\mathscr{CG}$ is cartesian closed there exists an outstanding internal functor $\mathbf{L}$ for the category of convenient vector spaces and continuous linear maps. Using this functor $\mathbf{L}$ it is immediately seen that the above "explicit" notion of (continuous) differentiability is the same as the corresponding "implicit" one, i.e. that there exist a continuous map $\mathrm{D}\alpha : U \to \mathbf{L}(E, F)$ such that for arbitrary $(x, y) \in U \sqcap E$ the equation

$$\lim_{0 \neq t \to 0} \frac{1}{t}\{\alpha(x + ty) - \alpha(x)\} = \mathrm{D}\alpha x(y)$$

holds. Put differently: "weak" and "strong" (continuous) differentiability coincide in our setting. Since banach and fréchet vector spaces are also convenient ones, we see that our calculus is a generalization of the classical fréchet calculus.

The main theorem of our differential calculus for convenient vector spaces then states that the category of convenient vector spaces and smooth maps is cartesian closed.

The article is divided into 5 sections:

Section 1: A review of the basic facts concerning compactly generated spaces.

Section 2: The general theory of compactly generated vector spaces and continuous linear maps.

Section 3: The development and the basic theorems for the differential calculus.

Section 4: The investigation of function spaces of differentiable and of smooth functions.

Section 5: A short look at smooth manifolds.

1. Topological background

In this section we state the basic facts concerning compactly generated spaces. For proofs we refer to [11].

We denote by $\mathscr{H}$ the category of hausdorff spaces and by $\mathscr{CG}$ the full subcategory of compactly generated spaces. We recall that a hausdorff space is called compactly generated if it carries the final topology with respect to the inclusions of its compact subspaces. If X is any hausdorff space we define the compactly generated space $\mathbf{CG}X$ to have the same points as X and to carry the final topology with respect to the inclusions of the compact subspaces of X. This association clearly defines a functor $\mathbf{CG} : \mathscr{H} \to \mathscr{CG}$.

Theorem 1.1. *The category $\mathscr{CG}$ is a complete and cocomplete full coreflective subcategory of $\mathscr{H}$ with $\mathbf{CG}: \mathscr{H} \to \mathscr{CG}$ as coreflector. The diagram*

$$\begin{tikzcd} \mathscr{H} \arrow[rr, "\mathbf{CG}"] \arrow[dr, "\mathbf{U}"'] & & \mathscr{CG} \arrow[dl, "\mathbf{U}"] \\ & \mathscr{E}\mathrm{ns} & \end{tikzcd}$$

commutes where $\mathbf{U}$ *denotes the usual forgetful functors.*

Proposition 1.2. *Let X be compactly generated and S a subset of X. If S is open or closed in X, the subspace topology on S is compactly generated.*

In order to avoid notational difficulties, we shall denote by the symbol $\times$ the usual (topological) product with respect to $\mathscr{H}$, whereas the symbol $\sqcap$ stands for the product with respect to $\mathscr{CG}$ calculated as $\sqcap = \mathbf{CG} \circ \times$.

From elementary topology we recall that the compact-open topology on function spaces defines an internal functor $\mathbf{CO}: \mathscr{H}^{\mathrm{op}} \times \mathscr{H} \to \mathscr{H}$. Hence

$$\mathbf{CG} \cdot \mathbf{CO} \underset{\mathrm{def}}{=} \mathbf{C}: \mathscr{CG}^{\mathrm{op}} \times \mathscr{CG} \to \mathscr{CG} \text{ is an internal functor for } \mathscr{CG}.$$

Theorem 1.3. (Gabriel–Zisman). *The category $\mathscr{CG}$ is cartesian closed with $\mathbf{C}: \mathscr{CG}^{\mathrm{op}} \times \mathscr{CG} \to \mathscr{CG}$ as internal functor. A function $\alpha: X \to \mathbf{C}(Y, Z)$ is continuous iff the corresponding function $\hat{\alpha}: X \sqcap Y \to Z$ with $\hat{\alpha}: (x, y) \mapsto \alpha x(y)$ is continuous.*

Theorem 1.4 (Steenrod). *The following diagram commutes*:

$$\begin{tikzcd} \mathscr{CG}^{\mathrm{op}} \times \mathscr{H} \arrow[r, "\mathbf{CO}"] \arrow[d, "1 \times \mathbf{CG}"] & \mathscr{H} \arrow[d, "\mathbf{CG}"] \\ \mathscr{CG}^{\mathrm{op}} \times \mathscr{CG} \arrow[r, "\mathbf{C}"] & \mathscr{CG} \end{tikzcd}$$

In order to exhibit the main relations between the category $\mathscr{CG}$ and the category $\mathscr{HU}$ of hausdorff uniform spaces we denote by $\mathbf{H}: \mathscr{HU} \to \mathscr{H}$ the usual "topologizing" functor. Further we denote by sc$\mathscr{HU}$ and c$\mathscr{HU}$ the full subcategories of $\mathscr{HU}$ with objects the sequentially complete resp. the complete hausdorff uniform spaces.

Proposition 1.5. *The categories $\mathscr{HU}$, sc$\mathscr{HU}$, c$\mathscr{HU}$ are complete and cocomplete.*

Next we remind the reader that for any hausdorff space X and any hausdorff uniform space Y the function space $\mathbf{CO}(X, \mathbf{H}Y)$ is uniformizable as follows: For C

compact in X and U an entourage of Y, the sets

$$(C, U) = \{(\alpha_1, \alpha_1) \in \mathbf{CO}(X, \mathbf{H}Y) \times \mathbf{CO}(X, \mathbf{H}Y) \mid (\alpha_1 x, \alpha_2 x) \in U \text{ for all } x \in C\}$$

form a subbase of a uniform structure for $\mathbf{CO}(X, \mathbf{H}Y)$. The resulting uniform space will be denoted by $\mathbf{CU}(X, Y)$ and the construction clearly extends to a functor $\mathbf{CU} : \mathcal{H}^{\mathrm{op}} \times \mathcal{HU} \to \mathcal{HU}$.

Proposition 1.6. *The following diagram commutes*:

$$\begin{tikzcd}
\mathcal{H}^{\mathrm{op}} \times \mathcal{HU} \arrow[r, "\mathbf{CU}"] \arrow[d, "1 \times \mathbf{H}"] & \mathcal{HU} \arrow[d, "\mathbf{H}"] \\
\mathcal{H}^{\mathrm{op}} \times \mathcal{H} \arrow[r, "\mathbf{CO}"] & \mathcal{H}
\end{tikzcd}$$

A very natural question now is whether or not the restriction of $\mathbf{CU}$ to $\mathcal{H}^{\mathrm{op}} \times \mathrm{c}\mathcal{HU}$ factors through $\mathrm{c}\mathcal{HU}$, and this question originally led Kelley to the definition of compactly generated spaces. We state:

Theorem 1.7 (Kelley). *The restriction of the functor* $\mathbf{CU}$ *to* $\mathcal{CG}^{\mathrm{op}} \times \mathrm{c}\mathcal{HU}$ *factors through* $\mathrm{c}\mathcal{HU}$ *and the same holds with respect to* $\mathrm{sc}\mathcal{HU}$. *Denoting these restrictions and factorizations also by* $\mathbf{CU}$, *the following diagram becomes commutative*:

$$\begin{tikzcd}
\mathcal{CG}^{\mathrm{op}} \times \mathrm{c}\mathcal{HU} \arrow[r, "\mathbf{CU}"] \arrow[d, hook] & \mathrm{c}\mathcal{HU} \arrow[d, hook] \\
\mathcal{CG}^{\mathrm{op}} \times \mathrm{sc}\mathcal{HU} \arrow[r, "\mathbf{CU}"] \arrow[d, "1 \times \mathbf{CG}\cdot\mathbf{H}"] & \mathrm{sc}\mathcal{HU} \arrow[d, "\mathbf{CG}\cdot\mathbf{H}"] \\
\mathcal{CG}^{\mathrm{op}} \times \mathcal{CG} \arrow[r, "\mathbf{C}"] & \mathcal{CG}.
\end{tikzcd}$$

2. Compactly generated real or complex vector spaces

First we state some basic facts concerning locally convex hausdorff topological vector spaces (for short: convex vector spaces) over the field $\mathbb{F}$ with $\mathbb{F}$ denoting the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. The category of these spaces with the continuous linear maps as arrows will be denoted by $\mathcal{LCV}$.

We observe that the compact-open topology gives us a functor $\mathbf{CO}:\mathscr{K}^{\text{op}}\times \mathscr{L}\mathscr{C}\mathscr{V}\to\mathscr{L}\mathscr{C}\mathscr{V}$ and analogously we obtain an internal functor $\mathbf{LCO}:\mathscr{L}\mathscr{C}\mathscr{V}^{\text{op}}\times \mathscr{L}\mathscr{C}\mathscr{V}\to\mathscr{L}\mathscr{C}\mathscr{V}$. Finally we see that n-linear continuous maps and the compact-open topology combine to give us for any $n\in\mathbb{N}$ corresponding functors $\mathbf{L}^n\mathbf{CO}:(\overset{n}{\times}\mathscr{L}\mathscr{C}\mathscr{V})^{\text{op}}\times\mathscr{L}\mathscr{C}\mathscr{V}\to\mathscr{L}\mathscr{C}\mathscr{V}$.

The main theorem for $\mathscr{L}\mathscr{C}\mathscr{V}$ is the theorem of Hahn–Banach. Together with other relevant properties one has:

Theorem 2.1 (Hahn–Banach). *The category $\mathscr{L}\mathscr{C}\mathscr{V}$ is complete, cocomplete, additive. The ground field $\mathbb{F}$ is a generator and a cogenerator. More precisely: If K is any closed convex subset of a convex vector space M and if $x\notin K$, then there exists a continuous linear map $\lambda:M\to\mathbb{F}$ such that $\lambda x\notin\overline{\lambda K}$.*

One says that a vector space E over $\mathbb{F}$ equipped with a compactly generated topology is a compactly generated vector space, if addition and scalar multiplication are continuous with respect to the $\mathscr{C}\mathscr{G}$-product $\sqcap$. We denote by $\mathscr{C}\mathscr{G}\mathscr{V}$ the corresponding category of compactly generated vector spaces and continuous linear maps.

Evidently we have a functor $\mathbf{CG}:\mathscr{L}\mathscr{C}\mathscr{V}\to\mathscr{C}\mathscr{G}\mathscr{V}$ and we denote by $\mathscr{C}\mathscr{G}\mathscr{V}^*$ the full subcategory of $\mathscr{C}\mathscr{G}\mathscr{V}$ generated by all compactly generated vector spaces of type $\mathbf{CG}M$ with M any convex vector space. Next we define a functor $\mathbf{LC}:\mathscr{C}\mathscr{G}\mathscr{V}^*\to\mathscr{L}\mathscr{C}\mathscr{V}$ as follows: If E is in $\mathscr{C}\mathscr{G}\mathscr{V}^*$, $\mathbf{LC}E$ has the same underlying vector space as E and carries the topology generated by the convex open subsets of E. Since $E=\mathbf{CG}M$ for some convex vector space M, the topology of $\mathbf{LC}E$, being finer than the topology of M, is hausdorff. One proves easily that addition and scalar multiplication remain continuous for $\mathbf{LC}E$ and the ordinary topological product.

Theorem 2.2. *The category $\mathscr{C}\mathscr{G}\mathscr{V}^*$ is complete, cocomplete, additive. The ground field $\mathbb{F}$ is a generator and a cogenerator. The functor $\mathbf{CG}:\mathscr{L}\mathscr{C}\mathscr{V}\to\mathscr{C}\mathscr{G}\mathscr{V}^*$ is left inverse and adjoint to the functor $\mathbf{LC}:\mathscr{C}\mathscr{G}\mathscr{V}^*\to\mathscr{L}\mathscr{C}\mathscr{V}$.*

Proof. We prove first that $\mathbf{CG}$ is left inverse and adjoint to $\mathbf{LC}$: If $E=\mathbf{CG}M$, we have already seen that $\mathbf{LC}E$ has a topology finer than M. Hence $\mathbf{CG}\circ\mathbf{LC}E$ has a topology finer than $\mathbf{CG}M=E$. On the other hand, the topology of E is certainly finer than the topology of $\mathbf{LC}E$, whence also finer than the topology of $\mathbf{CG}\circ\mathbf{LC}E$. It follows that $\mathbf{CG}$ is left inverse to $\mathbf{LC}$. To prove adjointness, let E be in $\mathscr{C}\mathscr{G}\mathscr{V}^*$, M in $\mathscr{L}\mathscr{C}\mathscr{V}$, and $\lambda:\mathbf{LC}E\to M$ a continuous linear map. Applying $\mathbf{CG}$ and observing that $\mathbf{CG}$ is left inverse to $\mathbf{LC}$, we see that $\lambda:E\to\mathbf{CG}M$ is also continuous. Conversely, if $\lambda:E\to\mathbf{CG}M$ is a continuous linear map, applying $\mathbf{LC}$ and observing that the topology of $\mathbf{LC}\circ\mathbf{CG}M$ is finer than the topology of M, we see that $\lambda:\mathbf{LC}E\to M$ is also continuous. Hence

$$\mathscr{L}\mathscr{C}\mathscr{V}(\mathbf{LC}E,M)=\mathscr{C}\mathscr{G}\mathscr{V}^*(E,\mathbf{CG}M)$$

which proves adjointness. From this follows immediately that the category $\mathscr{CGV}^*$ is isomorphic to the full reflective subcategory $\mathscr{LCV}^*$ of $\mathscr{LCV}$ whose objects are all convex vector spaces of type $\mathbf{LC} \circ \mathbf{CG}M$ with M any convex vector space. Since $\mathscr{LCV}$ is complete, cocomplete, additive, with $\mathbb{F}$ as a generator and a cogenerator, the same holds for $\mathscr{LCV}^*$ and hence also for $\mathscr{CGV}^*$.

If X is a compactly generated space and E a compactly generated vector space, we may and shall consider the function space $\mathbf{C}(X, E)$ of continuous maps with its associated compactly generated function space topology as a compactly generated vector space in the obvious way. Thus we obtain a corresponding functor $\mathbf{C}: \mathscr{CG}^{\mathrm{op}} \times \mathscr{CGV} \to \mathscr{CGV}$. Analogously we obtain an internal functor $\mathbf{L}: \mathscr{CGV}^{\mathrm{op}} \times \mathscr{CGV} \to \mathscr{CGV}$ if we define $\mathbf{L}(E, F)$ as the vector subspace of $\mathbf{C}(E, F)$ equipped with the subspace topology (which is compactly generated since $\mathbf{L}(E, F)$ is closed in $\mathbf{C}(E, F)$).

Lemma 2.3. *The restriction of the internal functor* $\mathbf{L}: \mathscr{CGV}^{\mathrm{op}} \times \mathscr{CGV} \to \mathscr{CGV}$ *to* $\mathscr{CGV}^{*\mathrm{op}} \times \mathscr{CGV}^*$ *factors through* $\mathscr{CGV}^*$. *Denote the resulting functor also by* $\mathbf{L}$. *Then the following diagram commutes*:

$$\begin{tikzcd}
\mathscr{CGV}^{*\mathrm{op}} \times \mathscr{CGV}^* \arrow[r, "\mathbf{L}"] \arrow[d, "\mathbf{LC}^{\mathrm{op}} \times \mathbf{LC}"] & \mathscr{CGV}^* \\
\mathscr{LCV}^{\mathrm{op}} \times \mathscr{LCV} \arrow[r, "\mathbf{LCO}"] & \mathscr{LCV} \arrow[u, "\mathbf{CG}"']
\end{tikzcd}$$

Proof. Let E and F be compactly generated vector space in $\mathscr{CGV}^*$. Since $\mathbf{CG}$ preserves induced structures, we have $\mathbf{L}(E, F) = \mathbf{CG} \circ \mathbf{LCO}(E, F)$. Since $\mathbf{CG} \circ \mathbf{LC}(F) = F$, Steenrod's Theorem 1.4 gives us

$$\mathbf{CG} \circ \mathbf{LCO}(E, F) = \mathbf{CG} \circ \mathbf{LCO}(E, \mathbf{LC}F).$$

Since $\mathbf{CG} \circ \mathbf{LC}E = E$, we obtain from Theorem 2.2 that $\mathbf{LCO}(E, \mathbf{LC}F) = \mathbf{LCO}(\mathbf{LC}E, \mathbf{LC}F)$. Hence

$$\mathbf{L}(E, F) = \mathbf{CG} \circ \mathbf{LCO}(\mathbf{LC}E, \mathbf{LC}F)$$

which proves everything stated.

Theorem 2.4. *The category* $\mathscr{CGV}^*$ *is a symmetric multiplicatively closed category with the ground field* $\mathbb{F}$ *as unit. The internal* (*closing*) *functor is* $\mathbf{L}$ *and the multiplicative functor* $\otimes$ *satisfies* $\mathbf{L}(E \otimes F, G) \approx \mathbf{L}^2(E, F; G)$ *with a natural isomorphism and the bilinear functor* $\mathbf{L}^2$ *defined analogously to* $\mathbf{L}$ *with respect to continuous bilinear maps.*

Proof. Since $\mathbf{C}(X, -): \mathscr{CG} \to \mathscr{CG}$ is limit preserving, the same holds for $\mathbf{L}(E, -): \mathscr{CGV}^* \to \mathscr{CGV}^*$ and $\mathbf{L}^2(E, F; -): \mathscr{CGV}^* \to \mathscr{CGV}^*$. Hence $\mathbf{L}^2(E, F; -)$ has a coadjoint functor $\mathbf{T}_{(E,F)}: \mathscr{CGV}^* \to \mathscr{CGV}^*$ by the special adjoint functor theorem. Defining $E \otimes F = \mathbf{T}_{(E,F)}\mathbb{F}$ we obtain the theorem by observing that clearly

$$\mathbf{L}(E, \mathbf{L}(F, G)) \approx \mathbf{L}^2(E, F; G),$$

$$\mathbf{L}^2(E, F; G) \approx \mathbf{L}^2(F, E; G), \qquad \mathbf{L}(\mathbb{F}, E) \approx E$$

with natural isomorphisms in all variables.

We define the dual E^* of a compactly generated vector space E by $E^* = \mathbf{L}(E, \mathbb{F})$. Using Theorem 2.4 we see immediately that the linear map $\sigma: E \to E^{**}$ with $\sigma x: \lambda \mapsto \lambda(x)$ is continuous. Since $\mathbb{F}$ is a cogenerator in $\mathscr{CGV}^*$, the map σ is injective whenever E is in $\mathscr{CGV}^*$. We say that E is embedded in its double dual space if $\sigma: E \to E^{**}$ is injective and E carries the induced compactly generated topology with respect to σ.

Theorem 2.5 (Frölicher–Jarchow). *Every compactly generated vector space of $\mathscr{CGV}^*$ is embedded in its double dual.*

Proof. Let E be in $\mathscr{CGV}^*$. It suffices to show that the σ-induced compactly generated topology on the vector space underlying E is finer than the topology of E. By Lemma 2.3 we have $E^* = \mathbf{CG} \circ \mathbf{LCO}(\mathbf{LC}E, \mathbb{F})$. Defining $E' = \mathbf{LCO}((\mathbf{LC}E, \mathbb{F})$ we have – again by Lemma 2.3 – that $E^{**} = \mathbf{CG} \circ \mathbf{LCO}(\mathbf{LC} \circ \mathbf{CG}E', \mathbb{F})$. Consider now the injective linear function $\sigma: \mathbf{LC}E \to \mathbf{LCO}(\mathbf{LC} \circ \mathbf{CG}E', \mathbb{F})$ and denote by E_σ the σ-induced convex vector space structure on the vector space underlying $\mathbf{LC}E$. If U is any closed convex circled zero neighborhood of $\mathbf{LC}E$, the polar U^0 of U is defined as

$$U^0 = \{\lambda \in E' \mid |\lambda x| \leq 1 \text{ for all } x \in U\}.$$

Since U is closed, convex, and circled, we obtain

$$\begin{aligned} U &= \{x \in \mathbf{LC}E \mid |\lambda x| \leq 1 \text{ for all } \lambda \in U^0\} \\ &= \sigma^{-1}\{f \in \mathbf{LCO}(\mathbf{LC} \circ \mathbf{CG}E', \mathbb{F}) \mid |f\lambda| \leq 1 \text{ for all } \lambda \in U^0\}. \end{aligned}$$

Hence U is a zero neighborhood in E_0 if U^0 is compact in $\mathbf{LC} \circ \mathbf{CG}E'$. Since U^0 is obviously an equicontinuous subset of E', the topology of U^0 is the same whether we consider U^0 as a subspace of E' or we equip U^0 with the topology of pointwise convergence. But in the latter case we obviously have a compact space. Hence U^0 is also compact in $\mathbf{LC} \circ \mathbf{CG}E'$. It follows that the topology of E_σ is finer than the topology of $\mathbf{LC}E$. Since the functor $\mathbf{CG}$ preserves induced topologies, $\mathbf{CG}E_\sigma$ has the compactly generated topology induced by $\sigma: E \to E^{**}$, and we conclude that this topology is finer than the topology of $\mathbf{CG} \circ \mathbf{LC}E = E$.

Because the objects of $\mathscr{LCV}$ are uniformizable, it is natural to investigate the full reflective subcategories of sequentially complete convex vector spaces $\mathrm{sc}\mathscr{LCV}$ and of complete convex vector spaces $c\mathcal{LCV}$. The restriction of the functor $\mathbf{CG}:\mathcal{LCV}\to\mathcal{CGV}$ to these subcategories then generates the full subcategories $sc\mathcal{CGV}$ and $c\mathcal{CGV}$ in the same way as the full subcategory $\mathcal{CGV}^{\#}$ has been created. Hence we may now restrict the functor $\mathbf{LC}:\mathcal{CGV}^{\#}\to\mathcal{LCV}$ to the subcategories $sc\mathcal{CGV}$ and $c\mathcal{CGV}$ and the question is: Do these restrictions factor again through $sc\mathcal{LCV}$, resp. $c\mathcal{LCV}$?

Theorem 2.6. *The restriction of the functor* $\mathbf{LC}:\mathcal{CGV}^{\#}\to\mathcal{LCV}$ *to the full subcategory* $sc\mathcal{CGV}$ *of* $\mathcal{CGV}^{\#}$ *factors through the full reflective subcategory* $sc\mathcal{LCV}$ *of* $\mathcal{LCV}$.

Proof. Let (x_n) be a cauchy sequence in $\mathbf{LC}E$, where $E=\mathbf{CG}M$ and M is a sequentially complete convex vector space. Since the topology of $\mathbf{LC}E$ is finer than the topology of M, the sequence (x_n) is also a cauchy sequence in M, hence convergent to some vector x_0 in M. The set $\{x_n \mid n\in\mathbb{N}\}\cup\{x_0\}$ is then compact in M and hence also compact in $E=\mathbf{CG}M$. Consequently (x_n) also converges in E to x_0. Because the topology of E is finer than the topology of $\mathbf{LC}E$, we finally see that the sequence (x_n) converges to x_0 in $\mathbf{LC}E$.

If we try to prove the analogue of Theorem 2.6 with respect to $c\mathcal{CGV}$ and $c\mathcal{LCV}$, we clearly run into problems. The ensuing difficulties cannot be simply circumvented by applying afterwards the completion reflector from $\mathcal{LCV}$ to $c\mathcal{LCV}$ because this functor generally enlarges the underlying vector spaces.

Restricting therefore our attention to $sc\mathcal{CGV}$, we deduce immediately from Lemma 2.3 and Kelley's Theorem 1.7 that the restriction of the internal functor $\mathbf{L}:\mathcal{CGV}^{\mathrm{op}}\times\mathcal{CGV}\to\mathcal{CGV}$ to $sc\mathcal{CGV}^{\mathrm{op}}\times sc\mathcal{CGV}$ factors through $sc\mathcal{CGV}$. Denoting the resulting functor also by $\mathbf{L}$, we obtain:

Theorem 2.7. *The category* $sc\mathcal{CGV}$ *is complete, cocomplete, additive. The ground field* $\mathbb{F}$ *is a generator and a cogenerator. The category* $sc\mathcal{CGV}$ *is symmetric multiplicatively closed with the ground field* $\mathbb{F}$ *as unit. The internal (closing) functor is* $\mathbf{L}:sc\mathcal{CGV}^{\mathrm{op}}\times sc\mathcal{CGV}\to sc\mathcal{CGV}$ *with* $\mathbf{L}(E,F)=\mathbf{CG}\circ\mathbf{LCO}(\mathbf{LC}E,\mathbf{LC}F)$. *The multiplicative functor* $\otimes$: $sc\mathcal{CGV}\times sc\mathcal{CGV}\to sc\mathcal{CGV}$ *satisfies* $\mathbf{L}(E\otimes F,G)\approx\mathbf{L}(E,\mathbf{L}(F,G))\approx\mathbf{L}^2(E,F;G)$ *with natural isomorphisms. Every object of* $sc\mathcal{CGV}$ *is embedded in its double dual.*

Denoting by $\mathcal{CGV}_{\mathrm{cont}}$ the category of compactly generated vector spaces with arrows the continuous maps, we get evident functors

$$\mathbf{C}:\mathcal{CG}^{\mathrm{op}}\times\mathcal{CGV}_{\mathrm{cont}}\to\mathcal{CGV}_{\mathrm{cont}}$$

and

$$\mathbf{C}:\mathcal{CGV}^{\mathrm{op}}_{\mathrm{cont}}\times\mathcal{CGV}_{\mathrm{cont}}\to\mathcal{CGV}_{\mathrm{cont}}.$$

Replacing $\mathcal{CGV}_{\mathrm{cont}}$ by $sc\mathcal{CGV}_{\mathrm{cont}}$ the restrictions of $\mathbf{C}$ to the respective full subcategories factor through $sc\mathcal{CGV}_{\mathrm{cont}}$. This follows immediately from Steenrod's Theorem 1.4 and Kelley's Theorem 1.7. Especially we have:

Theorem 2.8. *The category* $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$ *has arbitrary products which are calculated as for* $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$. *The category* $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$ *is cartesian closed with* $\mathbf{C}:\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}^{\mathrm{op}} \times \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}} \to \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$ *as internal functor. The natural homeomorphism* $\tau:\mathbf{C}(E,\mathbf{C}(F,G)) \to \mathbf{C}(E \sqcap F, G)$ *is linear. The ground field* $\mathbb{F}$ *is a generator and a cogenerator for* $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$.

3. Differential calculus for convenient vector spaces over the reals

From now on we shall fix the ground field to be the real numbers $\mathbb{R}$. Further we shall call a compactly generated vector space E a convenient vector space if E belongs to $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$, i.e. if E is of type $\mathbf{CG}M$ with M a sequentially complete convex vector space.

Definition 3.1. Let E and F be convenient vector spaces. Let U be open in E and V open in F, and let $\alpha: E \supset U \to V \subset F$ be a given function (defined on U with values in V). Then we call α *differentiable* (on U), if there exists a continuous map $\beta: E \supset U \to \mathbf{L}(E,F)$ such that for every fixed $(x,y) \in U \sqcap E$ we have: $\lim_{0\neq t\to 0}(1/t)\{\alpha(x+ty)-\alpha(x)\} = \beta x(y)$.

Proposition 3.2. *If* $\alpha: E \supset U \to V \subset F$ *is differentiable, the map* $\beta: U \to \mathbf{L}(E,F)$ *with* $\beta x(y) = \lim_{0\neq t\to 0}(1/t)\{\alpha(x+ty)-\alpha(x)\}$ *is unique.*

Definition 3.3. Let $\alpha: E \supset U \to V \subset F$ be differentiable. Then we define:

(i) The derivative $\mathrm{D}\alpha: U \to \mathbf{L}(E,F)$ by $\mathrm{D}\alpha x(y) = \lim_{0\neq t\to 0}(1/t)\{\alpha(x+ty)-\alpha(x)\}$,

(ii) The differential $\mathrm{d}\alpha: U \sqcap E \to F$ by $\mathrm{d}\alpha(x,y) = \lim_{0\neq t\to 0}(1/t)\{\alpha(x+ty)-\alpha(x)\}$,

(iii) The tangent $\mathbf{T}\alpha: U \sqcap E \to V \sqcap F$ by $\mathbf{T}\alpha(x,y) = (\alpha(x), \lim_{0\neq t\to 0}(1/t)\{\alpha(x+ty)+\alpha(x)\})$.

In each of these cases $(x,y) \in U \sqcap E$ is considered arbitrary but fixed, so that the limits exist.

In case the domain U of a differentiable map α is an open subset of $\mathbb{R}$, we denote by $\alpha': \mathbb{R} \supset U \to F$ the continuous map defined by $\alpha'(r) = \mathrm{D}\alpha r(1)$. We shall also use the notation $\mathrm{d}\alpha/\mathrm{d}r$ for α'.

Proposition 3.4. *If* $\alpha: \mathbb{R} \supset U \to V \subset F$ *is differentiable, then* α *is continuous.*

Proposition 3.5. *If* $\alpha: \mathbb{R} \supset U \to V \subset F$ *is differentiable and* $\lambda: F \to \mathbb{R}$ *is a continuous linear map, then* $\lambda \circ \alpha: \mathbb{R} \supset U \to \mathbb{R}$ *is differentiable and* $(\lambda \circ \alpha)' = \lambda \circ \alpha'$.

Now we shall state and prove the central theorem for differential calculus, often misleadingly called the mean value theorem. From this theorem all the important theorems of differential calculus follow.

Lemma 3.6 (The fundamental lemma of differential calculus). *Let $I=[0,1]$ be the closed unit interval in $\mathbb{R}$, let E be a convenient vector space, and let K be a closed convex subset of* **LC***E. If $\alpha:\mathbb{R}\supset I\to E$ is a continuous map such that its restriction to the open interval $(0,1)$ is differentiable and if $\alpha'(r)\in K$ for all $r\in(0,1)$, then $\alpha(1)-\alpha(0)\in K$.*

Proof. If $E=\mathbb{R}$ there exists an $r_0\in(0,1)$ such that $\alpha'(r_0)=\alpha(1)-\alpha(0)$ as everyone knows. Hence the lemma holds for $E=\mathbb{R}$.–Assume that the lemma does not hold for some $\alpha:I\to E$, i.e. $\alpha(1)-\alpha(0)\notin K$. By Hahn–Banach exists a continuous linear map $\lambda:\mathbf{LC}E\to\mathbb{R}$ such that $\lambda(\alpha(1)-\alpha(0))\notin\overline{\lambda K}$. Since $\mathbf{CG}\circ\mathbf{LC}E=E$, the linear map λ is also continuous as a map $\lambda:E\to\mathbb{R}$. By Proposition 3.5 we have $(\lambda\circ\alpha)'=\lambda\circ\alpha'$. Hence $(\lambda\circ\alpha)'(r)\in\overline{\lambda K}$ for all $r\in(0,1)$. Since the lemma holds for $\mathbb{R}$ we must have $\lambda\circ\alpha(1)-\lambda\circ\alpha(0)\in\overline{\lambda K}$. Contradiction!

The fundamental lemma at work

First we introduce a useful notation for differentiable maps: If $\alpha:E\supset U\to V\subset F$ is differentiable, we define

$$\theta R\alpha: E\sqcap E\sqcap\mathbb{R}\supset\{(x,y,t)\mid x\in U \text{ and } x+ty\in U\}\to F$$

by $\theta R\alpha(x,y,0)=0$ and for $t\neq 0$ we put

$$\theta R\alpha(x,y,t)=\frac{1}{t}\{\alpha(x+ty)-\alpha(x)\}-\mathrm{d}\alpha(x,y).$$

We observe that the domain of $\theta R\alpha$ is open in $E\sqcap E\sqcap\mathbb{R}$. We shall prove that $\theta R\alpha$ is continuous: First observe that for any $(x,y)\in U\sqcap E$ and any real numbers s and t with $x+sty\in U$ the formula

$$\theta R\alpha(x,sy,t)=s\theta R\alpha(x,y,st)$$

holds. Hence the function $\theta R\alpha(x,sy,t)$ is differentiable with respect to s for any fixed (x,y,t) and the derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta R\alpha(x,sy,t)=\mathrm{d}\alpha(x+tsy,y)-\mathrm{d}\alpha(x,y)\underset{\text{def}}{=}\gamma(x,y,t,s).$$

This function

$$\gamma: E\sqcap E\sqcap\mathbb{R}\sqcap\mathbb{R}\supset\{(x,y,t,s)\mid x\in U \text{ and } x+tsy\in U\}\to F$$

is continuous, has open domain, and satisfies $\gamma(x,y,t,0)=0$. Therefore: If $(x_0,y_0)\in U\sqcap E$ is fixed and K is any closed convex zero neighborhood in **LC**F, there exist positive real numbers δ and ε and a neighborhood $N_{(x_0,y_0)}$ of (x_0,y_0) in $U\sqcap E$, such that $\gamma(x,y,t,s)\in K$ whenever $|t|<\delta$, $|s|<\varepsilon$, $(x,y)\in N_{(x_0,y_0)}$. From the form of γ we infer that this implies $\gamma(x,y,t,s)\in K$ for $|t|<\delta\varepsilon$, $|s|\leq 1$, $(x,y)\in N_{(x_0,y_0)}$. By Proposition 3.4, $\theta R\alpha(x,sy,t)$ is continuous in s for $0\leq s\leq 1$ for any fixed t with $|t|<\delta\varepsilon$

and any fixed $(x, y) \in N_{(x_0, y_0)}$. Applying the fundamental lemma we get

$$\theta R\alpha(x, y, t) - \theta R\alpha(x, 0, t) = \theta R\alpha(x, y, t) \in K$$

for all $(x, y) \in N_{(x_0, y_0)}$ and $|t| < \delta\varepsilon$. This proves that

$$\theta R\alpha : E \sqcap E \sqcap \mathbb{R} \supset \{(x, y, t) | x \in U \quad \text{and} \quad x + ty \in U\} \to \mathbf{LC}F$$

is continuous at all points $(x, y, 0)$. Define $N_0 = \{y | (x_0, y) \in N_{(x_0, 0)}\}$ and $Z = \{ty | y \in N_0 \text{ and } |t| < \delta\varepsilon\}$. Hence Z is a zero neighborhood in E and for any $z \in Z$ we have $x_0 + z \in U$ and

$$\theta R\alpha(x_0, z, 1) = \alpha(x_0 + z) - \alpha(x_0) - \mathrm{d}\alpha(x_0, z) \in K.$$

This proves that $R\alpha : U \sqcap E \supset \{(x, y) | x \in U \text{ and } x + y \in U\} \to \mathbf{LC}F$, defined by

$$R\alpha(x, y) = \alpha(x + y) - \alpha(x) - \mathrm{d}\alpha(x, y),$$

is continuous at all points $(x, 0)$. Since $\mathrm{d}\alpha : U \sqcap E \to F$ is continuous, we obtain the continuity of $\alpha : E \supset U \to \mathbf{LC}F$ which in turn clearly implies the continuity of $\theta R\alpha$ as a function to $\mathbf{LC}F$ at all points (x, y, t) with $t \neq 0$. Hence $\theta R\alpha$ is continuous as a map to $\mathbf{LC}F$ and, since $F = \mathbf{CG} \circ \mathbf{LC}F$, we obtain also the continuity of $\theta R\alpha$ as a map to F.

Thus we have proved:

Theorem 3.7. *If a map $\alpha : E \supset U \to V \subset F$ is differentiable, then α is continuous.*

Theorem 3.8. *For a map $\alpha : E \supset U \to V \subset F$ are equivalent:*

(i) *α is differentiable,*

(ii) *there exists a continuous map $\beta : U \sqcap E \to F$ which is linear in the second variable, such that the map $\theta R\alpha : E \sqcap E \sqcap \mathbb{R} \supset \{(x, y, t) \mid x \in U$ and $x + ty \in U\} \to F$, defined by*

$$\theta R\alpha(x, y, t) = \frac{1}{t}\{\alpha(x + ty) - \alpha(x)\} - \beta(x, y)$$

for $t \neq 0$ and $\theta R\alpha(x, y, 0) = 0$, is continuous.

Since we work in a cartesian closed setting for continuous maps, it is clear that the continuity of the maps $\mathrm{D}\alpha : U \to \mathbf{L}(E, F)$, $\mathrm{d}\alpha : U \sqcap E \to F$, $\mathbf{T}\alpha : U \sqcap E \to V \sqcap F$ imply each other for any differentiable function $\alpha : E \supset U \to V \subset F$. Thus, so-called "weak" and "strong" differentiability are the same notions in our context. We shall make extensive use of this fact.

Proposition 3.9. *Let $\alpha : E \supset U \to V \subset F$ and $\beta : F \supset V \to W \subset G$ be differentiable. Then the composite map $\beta \circ \alpha : E \supset U \to W \subset G$ is differentiable.*

Theorem 3.10 (The chain rules). *Let $\alpha : E \supset U \to V \subset F$ and $\beta : F \supset V \to W \subset G$ be differentiable. Then the following formulas hold for the composite map:*

(i) $\mathrm{D}(\beta \circ \alpha) = \mathrm{comp} \circ \{\mathrm{D}\alpha, \mathrm{D}\beta \circ \alpha\} : E \supset U \to \mathbf{L}(E, F) \sqcap \mathbf{L}(F, G) \xrightarrow{\mathrm{comp}} \mathbf{L}(E, G)$,

(ii) $\mathrm{d}(\beta \circ \alpha) = \mathrm{d}\beta \circ (\alpha \sqcap \mathrm{d}\alpha) \circ (\Delta_U \sqcap 1_E) : E \sqcap E \supset U \sqcap E \to U \sqcap U \sqcap E \to V \sqcap F \to G$,

(iii) $\mathbf{T}(\beta \circ \alpha) = \mathbf{T}\beta \circ \mathbf{T}\alpha$.

The fact that only the tangent of differentiable mappings behaves functorially is the main reason why this explicit form of differentiation is in most cases more useful than the derivative. Observe that $\mathbf{T}(1_U)=1_{U\sqcap E}$ for the identity map $1_U: E\supset U\to U\subset E$. We put $\mathbf{T}U=U\sqcap E$ and call it the tangent space of $U\subset E$.

We note that constant maps, translations, continuous linear and multilinear maps are differentiable. Also finite sums and scalar multiples of differentiable maps are differentiable.

Theorem 3.11. *Let $\alpha: E\supset U\to V\subset F$ be differentiable. If U is connected in E, the following statements are equivalent*:

(i) *α is a constant map,*

(ii) $\mathrm{D}\alpha=0$.

Proof. (i)$\Rightarrow$(ii) is trivial. (ii)$\Rightarrow$(i): Let $x_0\in U$. Then $S=\{x\in U \mid \alpha x=\alpha x_0\}$ is clearly closed in U. For any $x\in S$ choose a radial open neighborhood $R(x)\subset U$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta R\alpha(x, sy, t)=0 \quad \text{for any } y\in R(x)-x,\ s\in I,\ t\in I,$$

we see that $\theta R\alpha(x, y, t)\in K$ for every $t\in[0,1]$ and any closed convex zero neighborhood K in $\mathbf{LC}F$. Hence $\theta R\alpha(x, y, 1)=\alpha(x+y)-\alpha(x)\in K$ for all closed convex zero neighborhoods K in $\mathbf{LC}F$. Hence $\alpha(x+y)=\alpha(x)$ whence the set S is open in U. The theorem follows.

Theorem 3.12 (Differentiable maps into a product). *Let $\alpha: E\supset U\to V\subset\sqcap_{i\in I}F_i$ be a map into a product space. Then the following statements are equivalent*:

(i) *α is differentiable,*

(ii) *for every $i\in I$ the map $\mathrm{pr}_i\circ\alpha: E\supset U\to V_i=\mathrm{pr}_i(V)\subset F_i$ is differentiable.*

This is clear. To prove an analogous statement for maps from a product, we have to restrict ourselves to the case of finite products.

Definition 3.13. Let $\alpha: E_1\sqcap E_2\supset U\to V\subset F$ be a function, where U is open in the convenient product vector space $E_1\sqcap E_2$ and V open in the convenient vector space F. We say that α is partially differentiable with respect to the first variable, if there exists a continuous map $\beta_1: U\to\mathbf{L}(E_1, F)$ such that for any fixed $(x, y_1)\in U\sqcap E_1$ we have: $\lim_{0\neq t\to 0}(1/t)\{\alpha(x+t(y_1,0))-\alpha(x)\}=\beta_1 x(y_1)$. – In an analogous way one defines partial differentiability with respect to the second variable and extends the definition to the case of any finite number of convenient factors E_i for $i>2$. – Finally, we say that an $\alpha:\sqcap_{i=1}^n E_i\supset U\to V\subset F$ is partially differentiable if it is partially differentiable with respect to all variables i with $1\leqslant i\leqslant n$.

Proposition 3.14. *If $\alpha:\sqcap_{i=1}^n E_i\supset U\to V\subset F$ is partially differentiable with respect to the jth variable, the corresponding $\beta_j: U\to\mathbf{L}(E_j, F)$ is unique.*

Definition 3.15. Let $\alpha : \sqcap_{i=1}^n E_i \supset U \to V \subset F$ be partially differentiable with respect to the jth variable. Then we define:

(i) the jth partial derivative $\mathrm{D}_j\alpha : \sqcap_{i=1}^n E_i \supset U \to \mathbf{L}(E_j, F)$ by

$$\mathrm{D}_j\alpha x(y_j) = \lim_{0\neq t\to 0} \frac{1}{t}\{\alpha(x+t(0,\ldots,0,y_j,0,\ldots,0)) - \alpha(x)\},$$

(ii) the jth partial differential $\mathrm{d}_j\alpha : (\sqcap_{i=1}^n E_i)\sqcap E_j \supset U\sqcap E_j \to F$ by

$$\mathrm{d}_j\alpha(x, y_j) = \mathrm{D}_j\alpha x(y_j),$$

(iii) the jth partial tangent $\mathbf{T}_j\alpha : (\sqcap_{i=1}^n E_i)\sqcap E_j \supset U\sqcap E_j \to V\sqcap F \subset F\sqcap F$ by

$$\mathbf{T}_j\alpha(x, y_j) = (\alpha(x), \mathrm{d}_j\alpha(x, y_j)).$$

Theorem 3.16 (Differentiable maps from a finite product). *Let $\alpha : \sqcap_{i=1}^n E_i \supset U \to V \subset F$ be a map out of a finite product space. Then the following statements are equivalent*:

(i) *α is differentiable,*

(ii) *α is partially differentiable.*

Proof. The implication (i)$\Rightarrow$(ii) is trivial. To see (ii)$\Rightarrow$(i) it clearly suffices to consider the case of a product $E_1\sqcap E_2$ of two factors. For fixed $(x, y_1, y_2) \in U\sqcap(E_1\sqcap E_2)$ and $t\neq 0$ we consider the function

$$\begin{aligned}
\theta R\alpha(x, y_1, y_2, t) &= \frac{1}{t}\{\alpha(x+t(y_1,y_2)) - \alpha(x)\} - \mathrm{d}_1\alpha(x, y_1) - \mathrm{d}_2\alpha(x, y_2)\\
&= \frac{1}{t}\{\alpha(x+t(y_1,y_2)) - \alpha(x+t(0,y_2))\} - \mathrm{d}_1\alpha(x+t(0,y_2), y_1)\\
&\quad + \frac{1}{t}\{\alpha(x+t(0,y_2)) - \alpha(x)\} - \mathrm{d}_2\alpha(x, y_2)\\
&\quad + \mathrm{d}_1\alpha(x+t(0,y_2), y_1) - \mathrm{d}_1\alpha(x, y_1)\\
&= \theta R_1\alpha(x+t(0,y_2), y_1, t) + \theta R_2\alpha(x, y_2, t)\\
&\quad + \mathrm{d}_1\alpha(x+t(0,y_2), y_1) - \mathrm{d}_1\alpha(x, y_1).
\end{aligned}$$

We are left to prove that $\lim_{0\neq t\to 0}\theta R_1\alpha(x+t(0,y_2), y_1, t) = 0$. To see this, consider the map $\theta R_1\alpha(x+t(0,y_2), sy_1, t)$ for $t\neq 0$ and $\theta R_1\alpha(x, sy_1, 0) = 0$. This map is differentiable with respect to s and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\theta R_1\alpha(x+t(0,y_2), sy_1, t) &= \mathrm{d}_1\alpha(x+t(0,y_2)+st(y_1,0), y_1)\\
&\quad - \mathrm{d}_1\alpha(x+t(0,y_2), y_1) \underset{\mathrm{def}}{=} \gamma(x, y_1, y_2, t, s).
\end{aligned}$$

The map γ is continuous with open domain and satisfying $\gamma(x, y_1, y_2, t, 0) = 0$.

Hence: For any closed convex zero neighborhood K of $\mathbf{LC}F$ exist positive real numbers δ and ε such that $\gamma(x, y_1, y_2, y, s) \in K$ whenever $|t| < \delta$ and $|s| < \varepsilon$. From the form of γ we infer that this implies $\gamma(x, y_1, y_2, t, s) \in K$ whenever $|t| < \mathrm{Min}\{\delta, \delta\varepsilon\}$ and $|s| \leq 1$. Application of the fundamental lemma gives $\theta R_1 \alpha(x + t(0, y_2), y_1, t) \in K$ for $|t| < \mathrm{Min}\{\delta, \delta\varepsilon\}$. Hence $\lim_{0 \neq t \to 0} \theta R_1 \alpha(x + t(0, y_2), y_1, t) = 0$.

As a corollary we obtain that our notion of differentiability coincides in the finite dimensional case with the usual notion of continuous differentiability.

Next we prove the decisive lemma which will allow us later to form convenient function spaces of differentiable mappings.

Lemma 3.17 (Convergence theorem for sequences of differentiable maps). *Let E and F be convenient vector spaces, and let U be open in E. Further suppose that (α_n) is a sequence of differentiable maps $\alpha_n : E \supset U \to F$. Then the following holds: If (α_n) is a cauchy sequence in $\mathbf{CO}(U, \mathbf{LC}F)$, and if $(\mathrm{d}\alpha_n)$ is a cauchy sequence in $\mathbf{CO}(U \sqcap E, \mathbf{LC}F)$, then the limit functions $\lim(\alpha_n) = \alpha : U \to F$ and $\lim(\mathrm{d}\alpha_n) = \beta : U \sqcap E \to F$ exist and are continuous. Moreover, α is differentiable and $\mathrm{d}\alpha = \beta$.*

Proof. Since $\mathbf{CG} \circ \mathbf{LC}F = F$, we obtain from Kelley's Theorem 1.7 the existence and the continuity of α and β. Clearly, β is linear in the second variable. If $(x, y) \in U \sqcap E$ is fixed, there exists an $\varepsilon > 0$ such that $x + ty \in U$ for all $|t| < \varepsilon$. Hence $\theta R\alpha_n(x, y, t) : \mathbb{R} \supset (-\varepsilon, \varepsilon) \to F$ is defined and continuous for all $n \in \mathbb{N}$. Because $\theta R\alpha_n$ is continuous, we get continuity of $\gamma_n : \mathbb{R} \sqcap \mathbb{R} \supset (-\varepsilon, \varepsilon) \sqcap I \to F$ with $\gamma_n(t, s) = \theta R\alpha_n(x, sy, t)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}s} \gamma_n(t, s) = \mathrm{d}\alpha_n(x + tsy, y) - \mathrm{d}\alpha_n(x, y).$$

Since the $\mathrm{d}\alpha_n$ form a cauchy sequence, we see that the sequence

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} \gamma_n\right) \in \mathbf{CO}((-\varepsilon, \varepsilon) \sqcap I, \mathbf{LC}F)$$

is also cauchy. Therefore, if C is compact in $(-\varepsilon, \varepsilon)$ and K is any closed convex zero neighborhood of $\mathbf{LC}F$, there exists a natural number N such that for all $m > N$ and $n > N$ we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} \gamma_m - \frac{\mathrm{d}}{\mathrm{d}s} \gamma_n\right)(C \sqcap I) \subset K.$$

Application of the fundamental lemma yields $(\theta R\alpha_m - \theta R\alpha_n)(x, y, t) \in K$ for all $t \in C$. Hence $(\theta R\alpha_n(x, y, -))$ is a cauchy sequence in $\mathbf{CO}((-\varepsilon, \varepsilon), \mathbf{LC}F)$. By Kelley's Theorem there exists a continuous $\rho : (-\varepsilon, \varepsilon) \to \mathbf{LC}F$ with $\rho = \lim(\theta R\alpha_n(x, y, -))$, and since $\mathbf{CG} \circ \mathbf{LC}F = F$ we may consider ρ as a continuous map $\rho : (-\varepsilon, \varepsilon) \to F$. For any fixed $t \neq 0$ we have

$$\rho(t) = \lim_{n\to\infty} \theta R\alpha_n(t) = \lim_{n\to\infty} \frac{1}{t}\{\alpha_n(x+ty) - \alpha_n(x)\} - \mathrm{d}\alpha_n(x, y)$$

$$= \frac{1}{t}\{\alpha(x+ty) - \alpha(x)\} - \beta(x, y)$$

and obviously $\rho(0) = 0$. Hence it follows that

$$\lim_{0\neq t\to 0} \frac{1}{t}\{\alpha(x+ty) - \alpha(x)\} = \beta(x, y)$$

which proves differentiability of α with differential $\mathrm{d}\alpha = \beta$.

Higher orders of differentiability are defined by iteration. So we have for example in the case of a 2-times differentiable map $\alpha: E \supset U \to V \subset F$ the first derivative $\mathrm{D}\alpha: U \to \mathbf{L}(E, F)$ and differentiating this map we obtain the second derivative $\mathrm{D}^2\alpha: U \to \mathbf{L}(E \otimes E, F)$. These derivatives are related to the second tangent

$$\mathbf{T}^2\alpha: U \sqcap E \sqcap E \sqcap E \to V \sqcap F \sqcap F \sqcap F$$

by the formula

$$\mathbf{T}^2\alpha(x, y, z_1, z_2) = (\alpha x, \mathrm{D}\alpha x(y), \mathrm{D}\alpha x(z_1), \mathrm{D}^2\alpha x(y \otimes z_1) + \mathrm{D}\alpha x(y_2)).$$

It is clear that the notions of higher order differentiability are equivalent whether defined by use of D or by use of $\mathbf{T}$. We already defined the tangent space of an open subset U of a convenient vector space E as $\mathbf{T}U = U \sqcap E$. We now define the tangent space of order n of U by $\mathbf{T}^n U = \mathbf{T}(\mathbf{T}^{n-1} U)$ for $n > 1$. With this notation we have for the i-th tangent of an n-times differentiable map $\alpha: U \to V$ that $\mathbf{T}^i\alpha: \mathbf{T}^i U \to \mathbf{T}^i V$ where $1 \leq i \leq n$.

Theorem 3.18 (Functoriality of the tangent operator). *Let $\alpha: E \supset U \to V \subset F$ and $\beta: F \supset V \to W \subset G$ be n-times differentiable. Then the composite map $\beta \circ \alpha: U \to W$ is n-times differentiable, and we obtain for the i-th tangent $\mathbf{T}^i(\beta \circ \alpha): \mathbf{T}^i U \to \mathbf{T}^i W$ that $\mathbf{T}^i(\beta \circ \alpha) = \mathbf{T}^i\beta \circ \mathbf{T}^i\alpha$ for $1 \leq i \leq n$.*

Clearly constant maps, translations, continuous linear and multilinear maps are n-times differentiable for arbitrary $n \in \mathbb{N}$.

The most important theorem on higher derivatives is:

Theorem 3.19 (Symmetry of higher derivatives). *If $\alpha: E \supset U \to V \subset F$ is n-times differentiable, then for each $x \in U$ the i-th derivative $\mathrm{D}^i\alpha x: \otimes^i E \to F$ is totally symmetric for every $i \in (1, \ldots, n)$.*

Proof. For $i = 1$ there is nothing to prove. Assume $i = 2$: Fix $(x, y_1, y_2) \in U \sqcap E \sqcap E$

and consider – whenever defined – the map

$$\gamma(t, s) = \theta R\alpha(x, sy_1 + y_2, t) - \theta R\alpha(x, sy_1, t) - \theta R\alpha(x, sy_2 + y_1, t) + \theta R\alpha(x, sy_2, t).$$

Clearly we can find a $\delta > 0$ such that $\gamma(t, s)$ is defined for $0 \leq s \leq 1$ and $|t| < \delta$. Differentiating with respect to s gives us:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\gamma(t, s) &= t\{\mathrm{D}^2\alpha x(y_2 \otimes y_1 - y_1 \otimes y_2)\\ &\quad + [\theta R\mathrm{D}\alpha(x, sy_1 + y_2, t) - \theta R\mathrm{D}\alpha(x, sy_1, t)](y_1)\\ &\quad - [\theta R\mathrm{D}\alpha(x, sy_2 + y_1, t) - \theta R\mathrm{D}\alpha(x, sy_2, t)](y_2)\}\\ &= t\{\mathrm{D}^2\alpha x(y_2 \otimes y_1 - y_1 \otimes y_2) + \sigma(t, s)\},\end{aligned}$$

where σ is continuous and satisfies $\sigma(0, s) = 0$. Since $[0, 1]$ is compact, there exists for any given closed convex zero neighborhood K of $\mathbf{LC}F$ an $\varepsilon > 0$ such that $\sigma(t, s) \in K$ for $0 \leq s \leq 1$ and $|t| < \varepsilon$. Let us assume that $\mathrm{D}^2\alpha x(y_2 \otimes y_1 - y_1 \otimes y_2) \neq 0$. Then select a closed convex zero neighborhood K such that $\mathrm{D}^2\alpha(y_1 \otimes y_2 - y_2 \otimes y_1) \notin K$ and choose $\varepsilon > 0$ such that $\sigma(t, s) \in K$ for $0 \leq s \leq 1$ and $|t| < \varepsilon$. Applying the fundamental lemma we get that

$$0 = \gamma(t, 1) - \gamma(t, 0) \in t\{\mathrm{D}^2\alpha x(y_2 \otimes y_1 - y_1 \otimes y_2) + K\} \quad \text{for } |t| < \varepsilon.$$

Contradiction! – By the inductive definition of higher differentiability it is now clear that $\mathrm{D}^i\alpha x$ is also totally symmetric for $i > 2$.

Now we prove the existence of primitive functions for a given continuous map $\alpha: \mathbb{R} \supset U \to E$. Here a map $\beta: \mathbb{R} \supset U \to E$ is called a primitive map for α if $\beta' = \alpha$.

Lemma 3.20. *Let E be a convenient vector space and $\alpha: \mathbb{R} \supset [0, 1] = I \to E$ a continuous map. Then there exists a continuous map $\beta: \mathbb{R} \supset I \to E$ which is differentiable on $(0, 1)$ with $\beta' = \alpha$ on the open interval $(0, 1)$.*

Proof. Subdivide I into 2^n parts of equal length and define $\alpha_n: I \to E$ by

$$\alpha_n(t) = \alpha(2^{-n}i) + (2^n t - i)\{\alpha(2^{-n}(i+1)) - \alpha(2^{-n}i)\}$$

for $i \leq 2^n t \leq i + 1$ and $i \in \{0, 1, \ldots, 2^{n-1}\}$. Further define $\beta_n: I \to E$ by

$$\begin{aligned}\beta_n(t) &= 2^{-1}(t - 2^{-n}i)[2\alpha(2^{-n}i) + (2^n t - i)\{\alpha(2^{-n}(i+1)) - \alpha(2^{-n}i)\}]\\ &\quad + 2^{-n-1}\sum_{j=0}^{i-1}\{\alpha(2^{-n}j) + \alpha(2^{-n}(j+1))\}\end{aligned}$$

for $i \leq 2^n t \leq i + 1$ and $i \in \{0, 1, \ldots, 2^{n-1}\}$. Obviously β_n is differentiable on $(0, 1)$ with $\beta_n' = \alpha_n$. Moreover $\beta_n(0) = 0$. Now we consider the sequences (α_n) and (β_n) in $\mathbf{CO}(I, \mathbf{LC}E)$. Since I is compact and α is continuous, there exists to any closed convex zero neighborhood K of $\mathbf{LC}E$ a natural number N such that $\alpha(t) - \alpha(t') \in K$

whenever $|t-t'|\leq 2^{-N}$. Hence $\alpha_n(t)-\alpha(t)\in IK+K$ whenever $n>N$. Consequently the sequence (α_n) is a cauchy sequence in $\mathbf{CO}(I, \mathbf{LC}E)$ converging to α. Now consider the maps $\gamma_{m,n}(t, s)=\beta_m(ts)-\beta_n(ts)$. These continuous maps satisfy $\gamma_{m,n}(t, 0)=0$, they are differentiable for any fixed t with respect to s, and

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma_{m,n}(t, s)=t\{\alpha_m(ts)-\alpha_n(ts)\}.$$

Since the (α_n) form a cauchy sequence, there exists for any closed convex zero neighborhood K a natural number N such that $(\alpha_m-\alpha_n)(I)\subset K$ whenever $m>N$ and $n>N$. But then we have $(\mathrm{d}/\mathrm{d}s)\gamma_{m,n}(t, s)\in tK$ whence application of the fundamental lemma yields $\gamma_{m,n}(t, 1)=(\beta_m-\beta_n)(t)\in tK$ whenever $m>N$ and $n>N$. From this we conclude that the sequence (β_n) is a cauchy sequence in $\mathbf{CO}(I, \mathbf{LC}E)$. Lemma 3.17 now tells us that the continuous map $\beta=\lim(\beta_n): I\to E$ is differentiable on $(0, 1)$ with $\beta'=\alpha$.

Theorem 3.21 (Existance of primitive maps). *Let E be a convenient vector space, let U be open in $\mathbb{R}$, and let $\alpha:\mathbb{R}\supset U\to E$ be a continuous map. Then there exists a primitive map $\beta:\mathbb{R}\supset U\to E$ for α. If U is connected (i.e. an open interval), the difference of any two primitive maps for α is constant.*

It is fairly simple to establish the main theorems of differential forms in this setting. For details we refer to [10]. First one defines the functor $\Lambda^n: \mathrm{sc}\mathscr{CGV}^{\mathrm{op}}\times \mathrm{sc}\mathscr{CGV}\to \mathrm{sc}\mathscr{CGV}$ of continuous n-linear alternating maps and then one considers for E, F convenient vector spaces and U open in E the graded vector space $\Omega(U, F)$ where $\Omega^n(U, F)$ is the vector space of smooth (infinitely differentiable) maps (differential forms) from U to $\Lambda^n(E, F)$. If $\omega\in\Omega^n(U, F)$ then $\delta^n\omega\in \Omega^{n+1}(U, F)$ is defined by

$$\delta^n\omega x(x_1, \ldots, x_{n+1})=\sum_{i=1}^{n+1}(-1)^{i-1}\,\mathrm{d}\omega(x, x_i)(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}).$$

Because second derivatives are symmetric one has $\delta\circ\delta=0$ and one obtains the cochain complex $(\Omega(U, F), \delta)$ of smooth differential forms.

Lemma 3.22 (Poincaré). *Let E and F be convenient vector spaces and U open in E. If $\omega\in\Omega(U, F)$ is a cycle ($\delta\omega=0$), then there exists for any point $x\in U$ an open x-neighborhood $U_x\subset U$ and an $\alpha\in\Omega(U_x, F)$ such that $\delta\alpha=\omega\mid U_x$.*

On the other hand one defines for U open in the convenient vector space E the graded vector space $C(U)$ of smooth singular cubes to be in dimension n the free vector space generated by the smooth maps $\gamma:\mathbb{R}^n\supset I^n\to U\subset E$ (clearly a map $\gamma: I^n\to U$ is called smooth if there exists a smooth map $\tilde{\gamma}:\mathbb{R}^n\supset V\to U\subset E$ with V open in $\mathbb{R}^n$, $I^n\subset V$, and $\tilde{\gamma}\mid I^n=\gamma$). Defining $\partial_n: C_n(U)\to C_{n-1}(U)$ as usual, we obtain the corresponding chain complex $(C(U), \partial)$ of smooth cubes. Finally one obtains the corresponding bilinear pairing $\int : \Omega^n(U,F)\otimes C_n(U)\to F$ for every $n\in\mathbb{N}$ and proves:

Theorem 3.23 (Stokes). *The following diagram commutes*:

$$\begin{tikzcd}
\Omega^{n-1}(U,F)\otimes C_n(U) \arrow[r, "1\otimes\partial_n"] \arrow[d, "\delta^{n-1}\otimes 1"'] & \Omega^{n-1}(U,F)\otimes C_{n-1}(U) \arrow[d, "\int"] \\
\Omega^n(U,F)\otimes C_n(U) \arrow[r, "\int"] & F
\end{tikzcd}$$

4. Convenient function spaces for differentiable and smooth maps

Let E be convenient vector spaces and let U be open in E. If α and β are n-times differentiable maps from U to F, and if r is any real number, we have

$$\mathrm{D}^n(r\alpha+\beta)=r\mathrm{D}^n\alpha+\mathrm{D}^n\alpha+\mathrm{D}^n\beta \quad\text{and}\quad \mathbf{T}^n(r\alpha+\beta)=r\mathbf{T}^n\alpha+\mathbf{T}^n\beta.$$

Hence the n-times differentiable maps from U to F form a vector space denoted by $\mathrm{D}^n(U,F)$. We extend this notion to the case $n=0$ by defining $\mathrm{D}^0(U,F)$ as the vector space of continuous maps from U to F. Since differentiability implies continuity, and higher differentiability is defined inductively, we have linear inclusions $i_{n,m}:\mathrm{D}^n(U,F)\hookrightarrow \mathrm{D}^m(U,F)$ whenever $0\leq m\leq n$. Hence the intersection $\bigcap_{n=0}^{\infty}\mathrm{D}^n(U,F)$ is a vector space, denoted by $\mathrm{D}^{\infty}(U,F)$ and called the vector space of smooth maps from U to F. Clearly $\mathrm{D}^{\infty}(U,F)=\lim \mathrm{D}^n(U,F)$ in the category of real vector spaces. Hence we have for every $n\in\mathbb{N}$ linear inclusions $i_n:\mathrm{D}^{\infty}(U,F)\hookrightarrow \mathrm{D}^n(U,F)$ satisfying the universal property associated with a limit.

We shall now turn the vector spaces $\mathrm{D}^n(U,F)$ and $\mathrm{D}^{\infty}(U,F)$ into convenient spaces as follows: For $n=0$ we provide $\mathrm{D}^0(U,F)$ with the convenient structure given by the functor $\mathbf{C}:\mathscr{C}\mathscr{G}^{\mathrm{op}}\times \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}\to \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$ described in Section 2. The resulting convenient vector space will henceforth be denoted as $\mathbf{C}^0(U,F)$. For $n=1$ we consider the linear injective function $\mathbf{T}_{\#}:\mathrm{D}^1(U,F)\to\mathbf{C}^0(\mathbf{T}U,\mathbf{T}F)$ defined by $\mathbf{T}_{\#}:\alpha\mapsto\mathbf{T}\alpha$. Then we induce on $\mathrm{D}^1(U,F)$ the compactly generated topology with respect to $\mathbf{T}_{\#}$ and denote the resulting compactly generated vector space by $\mathbf{C}^1(U,F)$. We prove that this is a convenient vector space: First we observe that $\mathbf{C}^0(\mathbf{T}U,\mathbf{T}F)$ is convenient and if $(\alpha_n)\in\mathbf{C}^1(U,F)$ is a sequence such that $(\mathbf{T}\alpha_n)$ is a cauchy sequence in $\mathbf{CO}(\mathbf{T}U,\mathbf{T}F)$, then we know by Lemma 3.17 that the sequence (α_n) converges to a differentiable map α with $\mathbf{T}\alpha=\lim(\mathbf{T}\alpha_n)$. For $n>1$ we induce on $\mathrm{D}^n(U,F)$ the compactly generated topology with respect to $\mathbf{T}^n_{\#}:\mathrm{D}^n(U,F)\to\mathbf{C}^0(\mathbf{T}^nU,\mathbf{T}^nF)$ where $\mathbf{T}^n_{\#}:\alpha\mapsto\mathbf{T}^n\alpha$. We denote the resulting convenient vector spaces by $\mathbf{C}^n(U,F)$.

By this definition it is clear that for $0 \leq m \leq n$ the linear inclusions $i_{n,m} : \mathbf{C}^n(U, F) \to \mathbf{C}^m(U, F)$ are continuous. Consequently, since $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$ is a complete category, the limit for the diagram

$$\{i_{n,m} : \mathbf{C}^n(U, F) \to \mathbf{C}^m(U, F) \mid n, m \in \mathbb{N} \text{ and } m \leq n\}$$

exists in $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$. The associated convenient limit vector space will be denoted by $\mathbf{S}(U, F)$. Clearly we may and shall choose $\mathrm{D}^\infty(U, F)$ as the vector space underlying $\mathbf{S}(U, F)$. We observe that $\mathbf{S}(U, F)$ may be identified as a subspace of the product space $\prod_{n=0}^\infty \mathbf{C}^n(U, F)$ in $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$, whence we obtain from Theorem 3.12 that a map $\alpha : G \supset W \to \mathbf{S}(U, F)$ is differentiable iff for all $n \in \mathbb{N}$ the maps $i_n \circ \alpha : W \to \mathbf{C}^n(U, F)$ are differentiable. And because Theorem 3.12 clearly extends to n-times differentiable and then to smooth maps, the construction of $\mathbf{S}(U, F)$ shows that an $\alpha : G \supset W \to \mathbf{S}(U, F)$ is smooth iff for all $n \in \mathbb{N}$ the maps $i_n \circ \alpha : W \to \mathbf{C}^n(U, F)$ are smooth. Thus we have:

Theorem 4.1. *Let E, F, G be convenient vector spaces, let U be open in E and W open in G. Then we have for each $n \in \mathbb{N}$ a convenient vector space $\mathbf{C}^n(U, F)$ with elements the n-times differentiable maps from U to F such that*

(i) $\mathbf{C}^0(U, F) = \mathbf{C}(U, F)$ *in the notation of Section* 2,

(ii) $\mathbf{C}^n(U, F)$ *has the compactly generated topology induced by the linear injective map* $\mathbf{T}^n_\# : \mathbf{C}(U, F) \to \mathbf{C}^0(\mathbf{T}^n U, \mathbf{T}^n F)$ *which is defined by* $\mathbf{T}^n_\#(\alpha) = \mathbf{T}^n\alpha$.

Hence for all $0 \leq j \leq n$ the linear injections $\mathbf{T}^j_\# : \mathbf{C}^n(U, F) \to \mathbf{C}^{n-j}(\mathbf{T}^j U, \mathbf{T}^j F)$ are continuous and $\mathbf{C}^n(U, F)$ has the compactly generated topology induced by each of these maps.

Further the linear inclusions $i_{n,m} : \mathbf{C}^n(U, F) \to \mathbf{C}^m(U, F)$ are continuous whenever $0 \leq m \leq n$.

Since the functor $\mathbf{C}(U, -) : \mathit{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}} \to \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$ is compatible with arbitrary products and by Theorem 3.12 the obvious linear map

$$\iota : \mathbf{C}^n(U, \prod_{i \in I} F_i) \to \prod_{i \in I} \mathbf{C}^n(U, F_i)$$

is a homeomorphism.

A map $\alpha : G \supset W \to \mathbf{C}^n(U, F)$ is j-times differentiable iff the map $\mathbf{T}^n_\# \circ \alpha : G \supset W \to \mathbf{C}^0(\mathbf{T}^n U, \mathbf{T}^n F)$ is j-times differentiable.

The convenient vector space $\mathbf{S}(U, F)$ of smooth maps from U to F is the intersection of the convenient vector spaces $\mathbf{C}^n(U, F)$ for all $n \in \mathbb{N}$ and is equipped with the corresponding compactly generated limit topology. Therefore we have continuous linear inclusions $i_n : \mathbf{S}(U, F) \to \mathbf{C}^n(U, F)$ for every $n \in \mathbb{N}$ and continuous linear injective maps $\mathbf{T}^n_\# : \mathbf{S}(U, F) \to \mathbf{S}(\mathbf{T}^n U, \mathbf{T}^n F)$. A map $\alpha : G \supset W \to \mathbf{S}(U, F)$ is n-times differentiable (or smooth) iff for every $n \in \mathbb{N}$ the map $i_n \circ \alpha = \alpha : G \supset W \to \mathbf{C}^n(U, F)$ is n-times differentiable (or smooth).

Finally we have for any product $\sqcap_{i\in I} F_i$ the obvious linear homeomorphism

$$\iota : \mathbf{S}(U, \prod_{i\in I} F_i) \to \prod_{i\in I} \mathbf{S}(U, F_i).$$

Lemma 4.2. *Let E, F, G be convenient vector spaces, let U be open in E, and let $\beta: F \to G$ be an n-times differentiable map where $n \geq 1$. Then β induces a differentiable map $\beta_* : \mathbf{C}^n(U, F) \to \mathbf{C}^{n-1}(U, G)$ defined by $\beta_*(\alpha) = \beta \circ \alpha$.*

Proof. First we show continuity of β_*: This is immediate from the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{C}^n(U, F) \arrow[r, "\mathbf{T}^n_*"] \arrow[d, "\beta_*"] & \mathbf{C}^0(\mathbf{T}^n U, \mathbf{T}^n F) \arrow[d, "(\mathbf{T}^n\beta)_*"] \\
\mathbf{C}^n(U, G) \arrow[r, "\mathbf{T}^n_*"] \arrow[d, "\iota_{n,n-1}"] & \mathbf{C}^0(\mathbf{T}^n U, \mathbf{T}^n G) \\
\mathbf{C}^{n-1}(U, G) &
\end{tikzcd}$$

Next we define $\mathbf{T}\beta_* : \mathbf{TC}^n(U, F) \to \mathbf{TC}^{n-1}(U, G)$ by commutativity of the upper side of the diagram

$$\begin{tikzcd}
\mathbf{TC}^n(U, F) = \mathbf{C}^n(U, F) \sqcap \mathbf{C}^n(U, F) \arrow[r, "\mathbf{T}\beta_*"] \arrow[d, "\iota^{-1}"] & \mathbf{TC}^{n-1}(U, G) = \mathbf{C}^{n-1}(U, G) \sqcap \mathbf{C}^{n-1}(U, G) \\
\mathbf{C}^n(U, \mathbf{T}F) \arrow[r, "(\mathbf{T}\beta)_*"] \arrow[d, "\iota_{n,n-1}"] & \mathbf{C}^{n-1}(U, \mathbf{T}G) \arrow[u, "\iota"] \arrow[dd, "\mathbf{T}^{n-1}_*"] \\
\mathbf{C}^{n-1}(U, \mathbf{T}F) \arrow[d, "\mathbf{T}^{n-1}_*"] & \\
\mathbf{C}^0(\mathbf{T}^{n-1} U, \mathbf{T}^n F) \arrow[r, "(\mathbf{T}^n\beta)_*"] & \mathbf{C}^0(\mathbf{T}^{n-1} U, \mathbf{T}^n G)
\end{tikzcd}$$

The lower side of this diagram clearly commutes. Hence $\mathbf{T}\beta_*$ – so defined – is continuous. The differential $\mathrm{d}\beta_*:\mathbf{TC}^n(U,F)\to\mathbf{C}^{n-1}(U,G)$, corresponding to $\mathbf{T}\beta_*$, is then given by $\mathrm{d}\beta_*(\alpha,\sigma)(x)=\mathrm{d}\beta(\alpha x,\sigma x)$, whence linear in the second variable. So we are left to prove that for fixed $(\alpha,\sigma)\in\mathbf{TC}^n(U,F)$ the map $\theta R\beta_*(\alpha,\sigma,-):\mathbb{R}\to\mathbf{C}^{n-1}(U,G)$, defined by

$$\theta R\beta_*(\alpha,\sigma,t)=\frac{1}{t}\{\beta_*(\alpha+t\sigma)-\beta_*(\alpha)\}-\mathrm{d}\beta_*(\alpha,\sigma)$$

for $t\neq 0$ and $\theta R\beta_*(\alpha,\sigma,0)=0$ is continuous for $t=0$. This is by definition of $\mathbf{C}^{n-1}(U,G)$ equivalent to continuity of $\mathbf{T}_*^{n-1}\circ\theta R\beta_*(\alpha,\sigma,-)$ and by cartesian closedness of $\mathbf{C}^0$ with respect to continuous maps this is equivalent to continuity of the associated explicit map $\Phi:\mathbf{T}^{n-1}U\sqcap\mathbb{R}\to\mathbf{T}^{n-1}G$ with

$$\Phi(\xi,t)=[\mathbf{T}_*^{n-1}\circ\theta R\beta_*(\alpha,\sigma,-)](\xi,t).$$

Since $\mathbf{T}^{n-1}$ defines and is defined by all the D^i for $0\leq i\leq n-1$, we have to show that for fixed $t\in\mathbb{R}$ the map $\theta R\beta_*(\alpha,\sigma,t):E\supset U\to G$ is $(n-1)$-times differentiable and the maps $\mathrm{D}^i\theta R\beta_*(\alpha,\sigma,t):U\to\mathbf{L}(\overset{i}{\otimes}E,G)$ are continuous in (x,t) with $\mathrm{D}^i\theta R\beta_*(\alpha,\sigma,0)=0$ for $0\leq i\leq n-1$. An easy computation gives us

$$\begin{aligned}
&\mathrm{D}^i\theta R\beta_*(\alpha,\sigma,t)(x)(x_1\otimes\cdots\otimes x_i)\\
&\quad=\theta R\mathrm{D}^i\beta(\alpha x,\sigma x,t)(\mathrm{d}[\alpha+t\sigma](x,x_1)\otimes\cdots\otimes\mathrm{d}[\alpha+t\sigma](x,x_i))\\
&\qquad+t\mathrm{D}^{i+1}\beta(\alpha x)(\mathrm{d}[\alpha+t\sigma](x,x_1)\otimes\cdots\otimes\mathrm{d}[\alpha+t\sigma](x,x_{i-1})\otimes\mathrm{d}\sigma(x,x_i)\otimes\sigma(x))\\
&\qquad+\sum_{k=1}^{i-1}\theta R\mathrm{D}^{i-1}\beta(\alpha x,\sigma x,t)(\mathrm{d}[\alpha+t\sigma](x,x_1)\otimes\cdots\otimes\mathrm{d}[\alpha+t\sigma](x,x_{k-1})\otimes\\
&\qquad\otimes\mathrm{d}^2[\alpha+t\sigma](x,x_k\otimes x_i)\otimes\mathrm{d}[\alpha+t\sigma](x,x_{k+1})\otimes\cdots\otimes\mathrm{d}[\alpha+t\sigma](x,x_{i-1}))\\
&\quad=\theta R\mathrm{D}^i\beta(\alpha x,\sigma x,t)(\mathrm{d}[\alpha+t\sigma](x,x_1)\otimes\cdots\otimes\mathrm{d}[\alpha+t\sigma](x,x_i))+\\
&\qquad+F_i((x,t),x_1,\ldots,x_i).
\end{aligned}$$

For $i=0$ we have $\theta R\beta_*(\alpha,\sigma,t)(x)=\theta R\beta(\alpha x,\sigma x,t)$. Hence $F_0=0$. By induction we see from our formula that $\theta R\beta_*(\alpha,\sigma,t):U\to G$ is $(n-1)$-times differentiable. By another induction we see that the F_i are $(n-1-i)$-times differentiable, are continuous in $((x,t),x_1,\ldots,x_i)$, and satisfy $F_i((x,0),x_1,\ldots,x_i)=0$. This proves the assertion.

Corollary 4.3. *Let E, F, G be convenient vector spaces, let U be open in E and $\beta:F\to G$ an n-times differentiable map. Then β induces i-times differentiable maps $\beta_*:\mathbf{C}^n(U,F)\to\mathbf{C}^{n-i}(U,G)$ for all $i\in(0,1,\ldots,n)$ which are defined by $\beta_*(\alpha)=\beta\circ\alpha$.*

Proof. For $i=0$ this is clear by commutativity of the diagram

$$\begin{tikzcd}
\mathbf{C}^n(U,F) \arrow[r, "\mathbf{T}^n_*"] \arrow[d, "\beta_*"] & \mathbf{C}^0(\mathbf{T}^nU,\mathbf{T}^nF) \arrow[d, "(\mathbf{T}^n\beta)_*"] \\
\mathbf{C}^n(U,G) \arrow[r, "\mathbf{T}^n_*"] & \mathbf{C}^0(\mathbf{T}^nU,\mathbf{T}^nG)
\end{tikzcd}$$

For $i=1$ this has been proved in Lemma 4.2. Suppose the corollary true for $j-1$ with $1 \leqslant j-1 < n$ and let $\beta: F \to G$ be j-times differentiable. Then $\beta_*: \mathbf{C}^n(U,F) \to \mathbf{C}^{n-j+1}(U,G)$ is $(j-1)$-times differentiable by hypothesis, whence $\beta_* = i_{n-j+1,n-j} \circ \beta_*: \mathbf{C}^n(U,F) \to \mathbf{C}^{n-j}(U,G)$ is $(j-1)$-times differentiable. The commutative diagram

$$\begin{tikzcd}
\mathbf{T}^{j-1}\mathbf{C}^n(U,F) \arrow[r, "\iota^{-1}"] \arrow[d, "\mathbf{T}^{j-1}\beta_*"] \arrow[dd, bend right=60, "\mathbf{T}^{j-1}\beta_*"'] & \mathbf{C}^n(U,\mathbf{T}^{j-1}F) \arrow[r, "i_{n,n-j+1}"] \arrow[d, "(\mathbf{T}^{j-1}\beta)_*"] & \mathbf{C}^{n-j+1}(U,\mathbf{T}^{j-1}F) \arrow[d, "(\mathbf{T}^{j-1}\beta)_*"] \\
\mathbf{T}^{j-1}\mathbf{C}^{n-j+1}(U,G) \arrow[r, "\iota^{-1}"] \arrow[d, "\mathbf{T}^{j-1}i_{n-j+1,n-j}"] & \mathbf{C}^{n-j+1}(U,\mathbf{T}^{j-1}G) \arrow[r, "i_{n-j+1,n-j}"] \arrow[d, "(\mathbf{T}^{j-1}1_G)_*"] & \mathbf{C}^{n-j}(U,\mathbf{T}^{j-1}G) \\
\mathbf{T}^{j-1}\mathbf{C}^{n-j}(U,G) \arrow[r, "\iota^{-1}"] & \mathbf{C}^{n-j}(U,\mathbf{T}^{j-1}G) \arrow[ur, equal] &
\end{tikzcd}$$

shows $\mathbf{T}^{j-1}\beta_*$ differentiable, since by Lemma 4.2 the far right map $(\mathbf{T}^{j-1}\beta)_*: \mathbf{C}^{n-j+1}(U,\mathbf{T}^{j-1}F) \to \mathbf{C}^{n-j}(U,\mathbf{T}^{j-1}G)$ is differentiable.

Lemma 4.4. *Let E, F, G be convenient vector spaces, let U be open in E and V open in F, and let $\alpha: E \supset U \to V \subset F$ be an n-times differentiable map. Then α induces a continuous linear map $\alpha^*: \mathbf{C}^n(V,G) \to \mathbf{C}^n(U,G)$, defined by $\alpha^*(\beta) = \beta \circ \alpha$.*

Proof. Clearly α^* is linear. The continuity follows from the commutative diagram

$$\begin{tikzcd}
\mathbf{C}^n(V,G) \arrow[r, "\mathbf{T}^n_*"] \arrow[d, "\alpha^*"] & \mathbf{C}^0(\mathbf{T}^nV,\mathbf{T}^nG) \arrow[d, "(\mathbf{T}^n\alpha)^*"] \\
\mathbf{C}^n(U,G) \arrow[r, "\mathbf{T}^n_*"] & \mathbf{C}^0(\mathbf{T}^nU,\mathbf{T}^nG)
\end{tikzcd}$$

Lemma 4.5. *Let E, F be convenient vector spaces and let U be open in E. If $n \geqslant 1$, the evaluation map $\varepsilon: \mathbf{C}^n(U,F) \sqcap U \to F$, defined by $\varepsilon(\alpha,x) = \alpha x$, is differentiable.*

Proof. Continuity of evaluation is clear from the commutative diagram

$$\begin{tikzcd} \mathbf{C}^n(U,F)\sqcap U \arrow[r, "i_{n,0}\sqcap 1_U"] \arrow[d, "\varepsilon"] & \mathbf{C}^0(U,F)\sqcap U \arrow[d] \\ F \arrow[r, equal] & F \end{tikzcd}$$

and cartesian closedness with respect to $\mathbf{C}^0$. Next we define maps $\lambda : \mathbf{T}(\mathbf{C}^n(U,F)\sqcap U)\to \mathbf{T}\mathbf{C}^n(U,F)\sqcap\mathbf{T}U$ by $\lambda(\alpha,x,\sigma,y)=((\alpha,\sigma),(x,y))$, $\pi_j:\mathbf{T}\mathbf{C}^n(U,F)\to\mathbf{C}^n(U,F)$ by $\pi_1(\alpha,\sigma)=\alpha$ and $\pi_2(\alpha,\sigma)=\sigma$, $\tau_j:\mathbf{T}F\to\mathbf{T}F$ by $\tau_1(y,z)=(y,z)$ and $\tau_2(y,z)=(0,y)$. Obviously all these maps are continuous linear maps. From these maps we obtain by composition for $j=1,2$ the maps $\rho_j=\tau_j\circ\varepsilon\circ(\mathbf{T}_*\sqcap 1_{\mathbf{T}U})\circ(\pi_j\sqcap 1_{\mathbf{T}U})\circ\lambda$ from $\mathbf{T}(\mathbf{C}^n(U,F)\sqcap U)$ to $\mathbf{T}F$. Since we know that $\varepsilon:\mathbf{C}^{n-1}(\mathbf{T}U,\mathbf{T}F)\sqcap\mathbf{T}U\to\mathbf{T}F$ is continuous, we deduce that $\rho_1+\rho_2$ is continuous. We shall prove that $\mathbf{T}\varepsilon=\rho_1+\rho_2$: To see this, we observe that the corresponding differential $\mathrm{d}\varepsilon$ satisfies $\mathrm{d}\varepsilon((\alpha,x),(\sigma,y))=\mathrm{d}\alpha(x,y)+\sigma(x)$, whence $\mathrm{d}\varepsilon$ is linear in the second variable. For fixed $((\alpha,x),(\sigma,y))\in\mathbf{T}(\mathbf{C}^n(U,F)\sqcap U)$ and $t\neq 0$ we have

$$\begin{aligned}\theta R\varepsilon((\alpha,x),(\sigma,y),t)&=\frac{1}{t}\{\varepsilon(\alpha+t\sigma,x+ty)-\varepsilon(\alpha,x)\}-\mathrm{d}\varepsilon((\alpha,x),(\sigma,y))\\&=\theta R(\alpha+t\sigma)(x,y,t)+t\,\mathrm{d}\sigma(x,y).\end{aligned}$$

Hence $\lim_{0\neq t\to 0}\theta R\varepsilon((\alpha,x),(\sigma,y),t)=0$.

Corollary 4.6. *Let E, F be convenient vector spaces and let U be open in E. Then the evaluation map $\varepsilon:\mathbf{C}^n(U,F)\sqcap U\to F$ is n-times differentiable.*

Proof. For $n=0$ we have continuity of ε by cartesian closedness with respect to continuous maps. For $n=1$ we have differentiability by Lemma 4.5. Suppose that the corollary is true for $n=k\geqslant 1$. Then we have for $n=k+1$ differentiability by Lemma 4.5 with derivative $\mathbf{T}\varepsilon=\rho_1+\rho_2$. The definition of the ρ_j involved – apart from $\varepsilon:\mathbf{C}^k(\mathbf{T}U,\mathbf{T}F)\sqcap\mathbf{T}U\to\mathbf{T}F$ – only continuous linear maps. By hypothesis the evaluation map $\varepsilon:\mathbf{C}^k(\mathbf{T}U,\mathbf{T}F)\sqcap\mathbf{T}U\to\mathbf{T}F$ is k-times differentiable. Hence $\mathbf{T}\varepsilon$ is k-times differentiable.

Lemma 4.7. *Let E, F be convenient vector spaces, let U be open in E and V open in F. Then the insertion map $\eta:E\supset U\to\mathbf{C}^n(F\supset V,E\sqcap F)$, defined by $\eta x:y\mapsto(x,y)$, is i-times differentiable for every $i\in\mathbb{N}$.*

Proof. We have $\eta=\kappa+\lambda$ where $\kappa:U\to\mathbf{C}^n(V,E\sqcap F)$ is defined by $\kappa x:y\mapsto(0,y)$, and where $\lambda:U\to\mathbf{C}^n(V,E\sqcap F)$ is defined by $\lambda x:y\mapsto(x,0)$. Hence η is the sum of the constant map κ and the linear map λ. We are left to prove the continuity of λ which is equivalent to the continuity of $\mathbf{T}^n_\# \circ \lambda : U \to \mathbf{C}^0(\mathbf{T}^n V, \mathbf{T}^n(E \sqcap F))$. Since

$$\mathbf{T}^n_\# \circ \lambda(x)(y_1, \ldots, y_{2^n}) = ((x, 0), (0, 0), \ldots, (0, 0)),$$

continuity follows.

Definition 4.8. We denote by $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$ the category with objects the convenient vector spaces and arrows the smooth maps.

We have the following fundamental theorem:

Theorem 4.9 (The fundamental theorem for smooth maps). *The category $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$ contains the category $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$ and is contained in the category $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$. All these categories have the same objects and the same (arbitrary) products. The category $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$ is cartesian closed with*

$$\mathbf{S} : \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}^{\mathrm{op}}_{\mathrm{smooth}} \times \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}} \to \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$$

as internal functor. This functor $\mathbf{S}$ is defined on objects as the limit of the diagram

$$\{\mathbf{C}^n(E, F) \xrightarrow{i_{n,n-1}} \mathbf{C}^{n-1}(E, F) \mid n \in \mathbb{N}\}$$

in $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}$, where $\mathbf{C}^0$ is the cartesian internal functor for $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{cont}}$ and $\mathbf{C}^n(E, F)$ has the compactly generated topology induced by the linear injective map $\mathbf{T}^n_\# : \mathbf{C}^n(E, F) \to \mathbf{C}^0(\mathbf{T}^n E, \mathbf{T}^n F)$. For arrows (i.e. smooth maps) the functor $\mathbf{S}$ is defined by composition.

The unit for cartesian closedness is given by the smooth insertion $\eta : E \to \mathbf{S}(F, E \sqcap F)$ with $\eta x : y \mapsto (x, y)$. The counit for cartesian closedness is given by the smooth evaluation $\varepsilon : \mathbf{S}(E, F) \sqcap E \to F$ with $\varepsilon(\alpha, x) = \alpha x$. A map $\alpha : E \to \mathbf{S}(F, G)$ is smooth iff the corresponding map $\hat{\alpha} = \varepsilon \circ (\alpha \sqcap 1) : E \sqcap F \to G$, defined by $\hat{\alpha}(x, y) = \alpha x(y)$, is smooth. The natural diffeomorphism $\sigma : \mathbf{S}(E, \mathbf{S}(F, G)) \to \mathbf{S}(E \sqcap F, G)$ is linear.

The ground field $\mathbb{R}$ is a generator and a cogenerator for $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$. The tangent functor $\mathbf{T} : \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}} \to \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$ with $\mathbf{T} : E \mapsto E \sqcap E$ and $\mathbf{T} : \alpha \mapsto \mathbf{T}\alpha$ with $\mathbf{T}\alpha(x, y) = (\alpha x, \mathrm{d}\alpha(x, y))$ is linear. Finally the diagram

$$\begin{tikzcd}
\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}^{\mathrm{op}}_{\mathrm{smooth}} \times \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}} \arrow[r, "\mathbf{S}"] \arrow[d, "1 \times \mathbf{T}"] & \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}} \arrow[d, "\mathbf{T}"] \\
\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}^{\mathrm{op}}_{\mathrm{smooth}} \times \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}} \arrow[r, "\mathbf{S}"] & \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}
\end{tikzcd}$$

commutes up to a smooth natural isomorphism.

Proof. 1. Functoriality of $\mathbf{S}$: Let $\alpha : E \to F$ be smooth. Then the following diagram commutes for all $n \in \mathbb{N}$:

$$\begin{tikzcd}
\mathbf{S}(F, G) \arrow[r, "i_n"] \arrow[d, "\mathbf{S}(\alpha, G) = \alpha^*"'] & \mathbf{C}^n(F, G) \arrow[d, "\alpha^*"] \\
\mathbf{S}(E, G) \arrow[r, "i_n"] & \mathbf{C}^n(E, G)
\end{tikzcd}$$

From Lemma 4.4 we know that $\alpha^* : \mathbf{C}^n(F, G) \to \mathbf{C}^n(E, G)$ is smooth for all $n \in \mathbb{N}$. Hence by the limit definition of $\mathbf{S}$ we see that $\alpha^* : \mathbf{S}(F, G) \to \mathbf{S}(E, G)$ is smooth. Let $\beta : F \to G$ be smooth. Then the following diagram commutes for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$:

$$\begin{tikzcd}
\mathbf{S}(E, F) \arrow[r, "i_{n+k}"] \arrow[d, "\mathbf{S}(E, \beta) = \beta_*"'] & \mathbf{C}^{n+k}(E, F) \arrow[d, "\beta_*"] \\
\mathbf{S}(E, G) \arrow[r, "i_n"] & \mathbf{C}^n(E, G)
\end{tikzcd}$$

The map $\beta_* : \mathbf{C}^{n+k}(E, F) \to \mathbf{C}^n(E, G)$ is k-times differentiable by Corollary 4.3. Hence by the limit definition of $\mathbf{S}$ we see that $\beta_* : \mathbf{S}(E, F) \to \mathbf{S}(E, G)$ is k-times differentiable. Since k is arbitrary we see that $\beta_* : \mathbf{S}(E, F) \to \mathbf{S}(E, G)$ is smooth.

2. Smoothness of the unit and counit: The insertion $\eta : E \to \mathbf{C}^n(F, E \sqcap F)$ is smooth for all $n \in \mathbb{N}$ by Lemma 4.7. Hence $\eta : E \to \mathbf{S}(F, E \sqcap F)$ is smooth by the limit definition of $\mathbf{S}$. The evaluation $\varepsilon : \mathbf{C}^n(E, F) \sqcap E \to F$ is n-times differentiable for every $n \in \mathbb{N}$ by Corollary 4.6. Since the diagram

$$\begin{tikzcd}
\mathbf{S}(E, F) \sqcap E \arrow[r, "\varepsilon"] \arrow[d, dash, "i_n \sqcap 1"'] & F \arrow[d, equal] \\
\mathbf{C}^n(E, F) \sqcap E \arrow[r, "\varepsilon"] & F
\end{tikzcd}$$

commutes, we see that $\varepsilon : \mathbf{S}(E, F) \sqcap E \to F$ is smooth.

It follows that $\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$ is cartesian closed. The other statements are now evident.

We add the following important remark: If we consider the category ${}_{\mathrm{open}}\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}$ with open subsets of convenient vector spaces as objects and smooth maps as arrows, then we obtain the more general functor

$$\mathbf{S} : {}_{\mathrm{open}}\mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}^{\mathrm{op}}_{\mathrm{smooth}} \times \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}} \to \mathrm{sc}\mathscr{C}\mathscr{G}\mathscr{V}_{\mathrm{smooth}}.$$

And as before we see that insertion $\eta : E \supset U \to \mathbf{S}(F \supset V, E \sqcap F)$ and evaluation $\varepsilon : \mathbf{S}(E \supset U, F) \sqcap U \to F$ are smooth maps, that a map $\alpha : E \supset U \to \mathbf{S}(F \supset V, G)$ is smooth iff the corresponding map $\hat{\alpha} = \varepsilon \circ (\alpha \sqcap 1) : E \sqcap F \supset U \sqcap V \to G$ is smooth, and that we have a smooth linear diffeomorphism $\sigma : \mathbf{S}(E \supset U, \mathbf{S}(F \supset V, G)) \to \mathbf{S}(E \sqcap F \supset U \sqcap V, G)$.

5. Other results, problems, differentiable manifolds

The first problem with respect to our setting concerns the inverse function theorem: One cannot expect that invertibility of the derivative at a point gives invertibility of the map in a neighborhood of this point. This can be seen by considering the map $\exp : \mathbf{S}(\mathbb{R}, \mathbb{R}) \to \mathbf{S}(\mathbb{R}, \mathbb{R})$ defined by $\exp \alpha(t) = \mathrm{e}^{\alpha(t)}$. Clearly $\mathrm{D}\exp(0) = 1_{\mathbf{S}(\mathbb{R},\mathbb{R})}$, but since $\exp(\alpha) : \mathbb{R} \to \mathbb{R}$ takes only positive values (whatever $\alpha : \mathbb{R} \to \mathbb{R}$ one starts with) and because every neighborhood of $\exp(0)$ clearly contains smooth maps also assuming negative values, invertibility of $\exp : \mathbf{S}(\mathbb{R}, \mathbb{R}) \to \mathbf{S}(\mathbb{R}, \mathbb{R})$ near the zero map is impossible. Therefore one has to impose additional conditions in order to obtain an inverse function theorem in our generalized setting.

Another interesting point concerns differential equations. By using suitable fixed points theorems for convenient vector spaces, one obtains the necessary tools for existence and uniqueness theorems for ordinary differential equations. Clearly these fixed point theorems are related to bounded subsets. More details can be found in [10].

Finally we shall touch some questions concerning differentiable manifolds. Instead of modelling manifolds on hilbert or banach spaces [8], we now model differentiable manifolds on convenient vector spaces with atlases consisting of differentiably compatible charts in the sense of our differential calculus. We then speak of convenient differentiable manifolds. Defining in the same spirit convenient differentiable vector bundles, we would like to obtain cartesian closedness of the category $\mathcal{M}_{\text{smooth}}$ of convenient smooth manifolds and smooth maps.

There are several obstructions: The first consists in defining charts for the set $S(M, N)$ of smooth maps from one convenient smooth manifold to another. What is clear however are the convenient vector spaces on which the "manifold" $S(M, N)$ has to be modelled: Near a smooth map $\alpha : M \to N$ the corresponding vector space has to be the vector space $S_\alpha(\tau_N)$ of smooth sections over α in the tangent bundle $\tau_N : \mathbf{T}N \to N$. More formally: $S_\alpha(\tau_N) = \{\sigma : M \to \mathbf{T}N \mid \tau_N \circ \sigma = \alpha,\ \sigma \text{ smooth}\}$. This fact has been noticed by Eells [3]. The convenient vector space topology of $S_\alpha(\tau_N)$ is now obtained as follows: For every $i \in \mathbb{N}$ we have $\mathbf{T}^i_{\#} : S_\alpha(\tau_N) \to \mathbf{C}(\mathbf{T}^i M, \mathbf{T}^{i+1} N)$ where $\mathbf{LC} \circ \mathbf{C}(\mathbf{T}^i M, \mathbf{T}^{i+1} N)$ clearly is a sequentially complete convex vector space. Hence $\mathbf{CG} \circ \mathbf{LC} \circ \mathbf{C}(\mathbf{T}^i M, \mathbf{T}^{i+1} N)$ is a convenient vector space and we take for $S_\alpha(\tau_N)$ the convenient compactly generated topology induced by the linear injective map

$$\{\mathbf{T}^i\} : S_\alpha(\tau_N) \to \prod_{i=0}^{\infty} \mathbf{CG} \circ \mathbf{LC} \circ \mathbf{C}(\mathbf{T}^i M, \mathbf{T}^{i+1} N).$$

So we know where the charts have to be situated but we lack the corresponding chart maps.

To obtain chart maps, I recall the following construction [8], possible for every smooth hilbert manifold N: There exists the so-called exponential map $\exp : \mathbf{T}N \supset U \to N$, defined on a neighborhood U of the zero section $0 : N \to \mathbf{T}N$, which together with the projection $\tau_N : \mathbf{T}N \to N$ gives a smooth diffeomorphism $\{\tau_N \mid U, \exp\} : U \to V \subset N \times N$ with an open neighborhood V of the diagonal in $N \times N$. Applying now Lemma II, 7.4 in [7] one can select a smaller neighborhood U' of the zero section in $\mathbf{T}N$ which is fiberwise smoothly diffeomorphic to $\mathbf{T}N$ itself. The two results together imply in the case of smooth hilbert manifolds the existence of a smooth map $\varepsilon : \mathbf{T}N \to N$ satisfying the following two conditions: (i) the composition $\varepsilon \circ 0$ of ε with the zero section is the identity map on N, and (ii) the smooth map $\{\tau_N, \varepsilon\} : \mathbf{T}N \to N \times N$ establishes a smooth diffeomorphism between $\mathbf{T}N$ and its image in $N \times N$ and this image is an open neighborhood of the diagonal in $N \times N$.

In the general case we take this situation as definition of what we call a smooth addition for a convenient smooth manifold.

Definition 5.1. Let N be a convenient smooth manifold. Then we say that a smooth map $\varepsilon_N : \mathbf{T}N \to N$ is a smooth addition for N if the following holds:

(i) If $0_N : N \to \mathbf{T}N$ is the zero section, then $\varepsilon_N \circ 0_N = 1_N : N \to N$, and

(ii) the map $\{\tau_N, \varepsilon_N\} : \mathbf{T}N \to N \times N$ is a smooth diffeomorphism between $\mathbf{T}N$ and its image in $N \times N$, and this image is open in $N \times N$.

Assuming that N is a convenient smooth manifold with a smooth addition $\varepsilon_N : \mathbf{T}N \to N$ we obtain for every $x \in N$ by restricting ε_N to the fiber $\tau_N^{-1}(x)$ over x a chart $\varepsilon_x = \varepsilon_N \mid \tau^{-1}(x) : \tau^{-1}(x) \to U_x \subset N$ near x, and these charts form a smooth atlas which defines the (original) smooth structure for N.

Now we consider $S(M, N)$ where M and N are convenient smooth manifolds and N is equipped with a smooth addition $\varepsilon_N : \mathbf{T}N \to N$. Then we obtain for every smooth $\alpha \in S(M, N)$ by composition with ε_N a smooth map $\varepsilon_\alpha : S_\alpha(\tau_N) \to S(M, N)$ defined by $\varepsilon_\alpha(\sigma) = \varepsilon \circ \sigma$. Clearly these maps ε_α are one-to-one for every α, whence they are the natural candidates for defining (induced) smooth charts for $S(M, N)$. The only problem is their smooth compatibility: They are compatible if M is a compact smooth manifold, and then the smooth structure of $S(M, N)$ obviously does not depend on the choice of the smooth addition ε_N for N.

Denoting by ${}_{\mathrm{comp}}\mathcal{M}_{\mathrm{smooth}}$ the category of compact smooth manifolds and smooth maps (which coincides with the category of compact convenient smooth manifolds) and denoting by ${}_{\mathrm{add}}\mathcal{M}_{\mathrm{smooth}}$ the category of convenient smooth manifolds with smooth addition, we obtain the following theorem:

Theorem 5.2. *There exists a functor*

$$\mathbf{S} : {}_{\mathrm{comp}}\mathcal{M}^{\mathrm{op}}_{\mathrm{smooth}} \times {}_{\mathrm{add}}\mathcal{M}_{\mathrm{smooth}} \to {}_{\mathrm{add}}\mathcal{M}_{\mathrm{smooth}}$$

defined on objects by $S(M, N)$ with smooth charts as above and defined on smooth maps by composition. This functor has the property that whenever M and N are compact smooth manifolds and P is a convenient smooth manifold with smooth addition, then the smooth manifolds $\mathbf{S}(M, \mathbf{S}(N, P))$ and $\mathbf{S}(M \sqcap N, P)$ are naturally diffeomorphic by the usual correspondence of $\alpha : M \to \mathbf{S}(N, P)$ with $\hat{\alpha} : M \sqcap N \to P$ where $\hat{\alpha}(x, y) = \alpha x(y)$.

The proof of Theorem 5.2 is straightforward, using of course the generalization of Theorem 4.9 at the end of Section 4 to show smoothness of induced maps. To obtain a smooth addition for $\mathbf{S}(M, N)$ one first establishes a natural smooth diffeomorphism $\delta : \mathbf{TS}(M, N) \to \mathbf{S}(M, \mathbf{T}N)$ analogous to the one in Theorem 4.9. The smooth addition $\varepsilon_N : \mathbf{T}N \to N$ then induces the smooth map $(\varepsilon_N)_* : \mathbf{S}(M, \mathbf{T}N) \to \mathbf{S}(M, N)$ and in $(\varepsilon_N)_* \circ \delta : \mathbf{TS}(M, N) \to \mathbf{S}(M, N)$ one obtains a smooth addition for $\mathbf{S}(M, N)$.

As we have shown, we did not obtain cartesian closedness for $\mathcal{M}_{\text{smooth}}$. What one should remove somehow is the compactness condition for M (and not the smooth addition $\varepsilon_N : \mathbf{T}N \to N$ for N) in order to obtain the smooth function space manifold $\mathbf{S}(M, N)$. But for this the topology on the section spaces $S_\alpha(\tau_N)$ is too coarse. If one now observes that this was also the reason why $\exp : \mathbf{S}(\mathbb{R}, \mathbb{R}) \to \mathbf{S}(\mathbb{R}, \mathbb{R})$ is not invertible near the zero map and why a countable infinite product of one-dimensional spheres is not any more a manifold, the conjecture comes that it may be necessary to model manifolds (and to develop differential calculus) not with respect to open subsets of convenient vector spaces but with respect to – possible – smaller ones.

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